LEARNING OBJECTIVES

In this section, you will:

- Simplify rational expressions.
- Multiply rational expressions.
- Divide rational expressions.
- Add and subtract rational expressions.
- Simplify complex rational expressions.

1.6 RATIONAL EXPRESSIONS

A pastry shop has fixed costs of \$280 per week and variable costs of \$9 per box of pastries. The shop's costs per week in terms of x, the number of boxes made, is $280 + 9x$. We can divide the costs per week by the number of boxes made to determine the cost per box of pastries.

$$\frac{280 + 9x}{x}$$

Notice that the result is a polynomial expression divided by a second polynomial expression. In this section, we will explore quotients of polynomial expressions.

Simplifying Rational Expressions

The quotient of two polynomial expressions is called a **rational expression**. We can apply the properties of fractions to rational expressions, such as simplifying the expressions by canceling common factors from the numerator and the denominator. To do this, we first need to factor both the numerator and denominator. Let's start with the rational expression shown.

$$\frac{x^2 + 8x + 16}{x^2 + 11x + 28}$$

We can factor the numerator and denominator to rewrite the expression.

$$\frac{(x + 4)^2}{(x + 4)(x + 7)}$$

Then we can simplify that expression by canceling the common factor $(x + 4)$.

$$\frac{x + 4}{x + 7}$$

How To...

Given a rational expression, simplify it.

1. Factor the numerator and denominator.
2. Cancel any common factors.

Example 1 Simplifying Rational Expressions

Simplify $\frac{x^2 - 9}{x^2 + 4x + 3}$.

Solution

$\frac{(x + 3)(x - 3)}{(x + 3)(x + 1)}$ Factor the numerator and the denominator.

$\frac{x - 3}{x + 1}$ Cancel common factor $(x + 3)$.

Analysis *We can cancel the common factor because any expression divided by itself is equal to* 1.

Q & A...

Can the x^2 term be cancelled in Example 1?

No. A factor is an expression that is multiplied by another expression. The x^2 term is not a factor of the numerator or the denominator.

Try It #1

Simplify $\frac{x-6}{x^2-36}$.

Multiplying Rational Expressions

Multiplication of rational expressions works the same way as multiplication of any other fractions. We multiply the numerators to find the numerator of the product, and then multiply the denominators to find the denominator of the product. Before multiplying, it is helpful to factor the numerators and denominators just as we did when simplifying rational expressions. We are often able to simplify the product of rational expressions.

How To...

Given two rational expressions, multiply them.

1. Factor the numerator and denominator.
2. Multiply the numerators.
3. Multiply the denominators.
4. Simplify.

Example 2 **Multiplying Rational Expressions**

Multiply the rational expressions and show the product in simplest form:

$$\frac{x^2+4x-5}{3x+18} \cdot \frac{2x-1}{x+5}$$

Solution

$$\frac{(x+5)(x-1)}{3(x+6)} \cdot \frac{(2x-1)}{(x+5)}$$ Factor the numerator and denominator.

$$\frac{(x+5)(x-1)(2x-1)}{3(x+6)(x+5)}$$ Multiply numerators and denominators.

$$\frac{\cancel{(x+5)}(x-1)(2x-1)}{3(x+6)\cancel{(x+5)}}$$ Cancel common factors to simplify.

$$\frac{(x-1)(2x-1)}{3(x+6)}$$

Try It #2

Multiply the rational expressions and show the product in simplest form:

$$\frac{x^2+11x+30}{x^2+5x+6} \cdot \frac{x^2+7x+12}{x^2+8x+16}$$

Dividing Rational Expressions

Division of rational expressions works the same way as division of other fractions. To divide a rational expression by another rational expression, multiply the first expression by the reciprocal of the second. Using this approach, we would rewrite $\frac{1}{x} \div \frac{x^2}{3}$ as the product $\frac{1}{x} \cdot \frac{3}{x^2}$. Once the division expression has been rewritten as a multiplication expression, we can multiply as we did before.

$$\frac{1}{x} \cdot \frac{3}{x^2} = \frac{3}{x^3}$$

How To...

Given two rational expressions, divide them.

1. Rewrite as the first rational expression multiplied by the reciprocal of the second.
2. Factor the numerators and denominators.
3. Multiply the numerators.
4. Multiply the denominators.
5. Simplify.

Example 3 **Dividing Rational Expressions**

Divide the rational expressions and express the quotient in simplest form:

$$\frac{2x^2 + x - 6}{x^2 - 1} \div \frac{x^2 - 4}{x^2 + 2x + 1}$$

Solution

$$\frac{2x^2 + x - 6}{x^2 - 1} \cdot \frac{x^2 + 2x + 1}{x^2 - 4}$$ Rewrite as multiplication.

$$\frac{(2x - 3)(x + 2)}{(x + 1)(x - 1)} \cdot \frac{(x + 1)(x + 1)}{(x + 2)(x - 2)}$$ Factor the numerator and denominator.

$$\frac{(2x - 3)(x + 2)(x + 1)(x + 1)}{(x + 1)(x - 1)(x + 2)(x - 2)}$$ Multiply numerators and denominators.

$$\frac{(2x - 3)\cancel{(x + 2)}\cancel{(x + 1)}(x + 1)}{\cancel{(x + 1)}(x - 1)\cancel{(x + 2)}(x - 2)}$$ Cancel common factors to simplify.

$$\frac{(2x - 3)(x + 1)}{(x - 1)(x - 2)}$$

Try It #3

Divide the rational expressions and express the quotient in simplest form:

$$\frac{9x^2 - 16}{3x^2 + 17x - 28} \div \frac{3x^2 - 2x - 8}{x^2 + 5x - 14}$$

Adding and Subtracting Rational Expressions

Adding and subtracting rational expressions works just like adding and subtracting numerical fractions. To add fractions, we need to find a common denominator. Let's look at an example of fraction addition.

$$\frac{5}{24} + \frac{1}{40} = \frac{25}{120} + \frac{3}{120}$$
$$= \frac{28}{120}$$
$$= \frac{7}{30}$$

We have to rewrite the fractions so they share a common denominator before we are able to add.We must do the same thing when adding or subtracting rational expressions.

The easiest common denominator to use will be the **least common denominator**, or LCD. The LCD is the smallest multiple that the denominators have in common. To find the LCD of two rational expressions, we factor the expressions and multiply all of the distinct factors. For instance, if the factored denominators were $(x + 3)(x + 4)$ and $(x + 4)(x + 5)$, then the LCD would be $(x + 3)(x + 4)(x + 5)$.

Once we find the LCD, we need to multiply each expression by the form of 1 that will change the denominator to the LCD. We would need to multiply the expression with a denominator of $(x + 3)(x + 4)$ by $\frac{x + 5}{x + 5}$ and the expression with a denominator of $(x + 3)(x + 4)$ by $\frac{x + 3}{x + 3}$.

How To...

Given two rational expressions, add or subtract them.

1. Factor the numerator and denominator.
2. Find the LCD of the expressions.
3. Multiply the expressions by a form of 1 that changes the denominators to the LCD.
4. Add or subtract the numerators.
5. Simplify.

Example 4 **Adding Rational Expressions**

Add the rational expressions:

$$\frac{5}{x} + \frac{6}{y}$$

Solution First, we have to find the LCD. In this case, the LCD will be xy. We then multiply each expression by the appropriate form of 1 to obtain xy as the denominator for each fraction.

$$\frac{5}{x} \cdot \frac{y}{y} + \frac{6}{y} \cdot \frac{x}{x}$$

$$\frac{5y}{xy} + \frac{6x}{xy}$$

Now that the expressions have the same denominator, we simply add the numerators to find the sum.

$$\frac{6x + 5y}{xy}$$

Analysis *Multiplying by* $\frac{y}{y}$ *or* $\frac{x}{x}$ *does not change the value of the original expression because any number divided by itself is* 1*, and multiplying an expression by* 1 *gives the original expression.*

Example 5 **Subtracting Rational Expressions**

Subtract the rational expressions:

$$\frac{6}{x^2 + 4x + 4} - \frac{2}{x^2 - 4}$$

Solution

$$\frac{6}{(x+2)^2} - \frac{2}{(x+2)(x-2)}$$ Factor.

$$\frac{6}{(x+2)^2} \cdot \frac{x-2}{x-2} - \frac{2}{(x+2)(x-2)} \cdot \frac{x+2}{x+2}$$ Multiply each fraction to get LCD as denominator.

$$\frac{6(x-2)}{(x+2)^2(x-2)} - \frac{2(x+2)}{(x+2)^2(x-2)}$$ Multiply.

$$\frac{6x - 12 - (2x + 4)}{(x+2)^2(x-2)}$$ Apply distributive property.

$$\frac{4x - 16}{(x+2)^2(x-2)}$$ Subtract.

$$\frac{4(x-4)}{(x+2)^2(x-2)}$$ Simplify.

Q & A...

Do we have to use the LCD to add or subtract rational expressions?

No. Any common denominator will work, but it is easiest to use the LCD.

Try It #4

Subtract the rational expressions: $\dfrac{3}{x+5} - \dfrac{1}{x-3}$.

Simplifying Complex Rational Expressions

A complex rational expression is a rational expression that contains additional rational expressions in the numerator, the denominator, or both. We can simplify complex rational expressions by rewriting the numerator and denominator as single rational expressions and dividing. The complex rational expression $\dfrac{a}{\frac{1}{b}+c}$ can be simplified by rewriting the numerator as the fraction $\dfrac{a}{1}$ and combining the expressions in the denominator as $\dfrac{1+bc}{b}$. We can then rewrite the expression as a multiplication problem using the reciprocal of the denominator. We get $\dfrac{a}{1} \cdot \dfrac{b}{1+bc}$, which is equal to $\dfrac{ab}{1+bc}$.

How To…

Given a complex rational expression, simplify it.

1. Combine the expressions in the numerator into a single rational expression by adding or subtracting.
2. Combine the expressions in the denominator into a single rational expression by adding or subtracting.
3. Rewrite as the numerator divided by the denominator.
4. Rewrite as multiplication.
5. Multiply.
6. Simplify.

Example 6 Simplifying Complex Rational Expressions

Simplify: $\dfrac{y+\frac{1}{x}}{\frac{x}{y}}$.

Solution Begin by combining the expressions in the numerator into one expression.

$y \cdot \dfrac{x}{x} + \dfrac{1}{x}$ Multiply by $\dfrac{x}{x}$ to get LCD as denominator.

$\dfrac{xy}{x} + \dfrac{1}{x}$

$\dfrac{xy+1}{x}$ Add numerators.

Now the numerator is a single rational expression and the denominator is a single rational expression.

$$\dfrac{\frac{xy+1}{x}}{\frac{x}{y}}$$

We can rewrite this as division, and then multiplication.

$\dfrac{xy+1}{x} \div \dfrac{x}{y}$

$\dfrac{xy+1}{x} \cdot \dfrac{y}{x}$ Rewrite as multiplication.

$\dfrac{y(xy+1)}{x^2}$ Multiply.

Try It #5

Simplify: $\dfrac{\frac{x}{y} - \frac{y}{x}}{y}$

Q & A…

Can a complex rational expression always be simplified?

Yes. We can always rewrite a complex rational expression as a simplified rational expression.

Access these online resources for additional instruction and practice with rational expressions.

- Simplify Rational Expressions (http://openstaxcollege.org/l/simpratexpress)
- Multiply and Divide Rational Expressions (http://openstaxcollege.org/l/multdivratex)
- Add and Subtract Rational Expressions (http://openstaxcollege.org/l/addsubratex)
- Simplify a Complex Fraction (http://openstaxcollege.org/l/complexfract)

1.6 SECTION EXERCISES

VERBAL

1. How can you use factoring to simplify rational expressions?

2. How do you use the LCD to combine two rational expressions?

3. Tell whether the following statement is true or false and explain why: You only need to fi d the LCD when adding or subtracting rational expressions.

ALGEBRAIC

For the following exercises, simplify the rational expressions.

4. $\dfrac{x^2-16}{x^2-5x+4}$

5. $\dfrac{y^2+10y+25}{y^2+11y+30}$

6. $\dfrac{6a^2-24a+24}{6a^2-24}$

7. $\dfrac{9b^2+18b+9}{3b+3}$

8. $\dfrac{m-12}{m^2-144}$

9. $\dfrac{2x^2+7x-4}{4x^2+2x-2}$

10. $\dfrac{6x^2+5x-4}{3x^2+19x+20}$

11. $\dfrac{a^2+9a+18}{a^2+3a-18}$

12. $\dfrac{3c^2+25c-18}{3c^2-23c+14}$

13. $\dfrac{12n^2-29n-8}{28n^2-5n-3}$

For the following exercises, multiply the rational expressions and express the product in simplest form.

14. $\dfrac{x^2-x-6}{2x^2+x-6}\cdot\dfrac{2x^2+7x-15}{x^2-9}$

15. $\dfrac{c^2+2c-24}{c^2+12c+36}\cdot\dfrac{c^2-10c+24}{c^2-8c+16}$

16. $\dfrac{2d^2+9d-35}{d^2+10d+21}\cdot\dfrac{3d^2+2d-21}{3d^2+14d-49}$

17. $\dfrac{10h^2-9h-9}{2h^2-19h+24}\cdot\dfrac{h^2-16h+64}{5h^2-37h-24}$

18. $\dfrac{6b^2+13b+6}{4b^2-9}\cdot\dfrac{6b^2+31b-30}{18b^2-3b-10}$

19. $\dfrac{2d^2+15d+25}{4d^2-25}\cdot\dfrac{2d^2-15d+25}{25d^2-1}$

20. $\dfrac{6x^2-5x-50}{15x^2-44x-20}\cdot\dfrac{20x^2-7x-6}{2x^2+9x+10}$

21. $\dfrac{t^2-1}{t^2+4t+3}\cdot\dfrac{t^2+2t-15}{t^2-4t+3}$

22. $\dfrac{2n^2-n-15}{6n^2+13n-5}\cdot\dfrac{12n^2-13n+3}{4n^2-15n+9}$

23. $\dfrac{36x^2-25}{6x^2+65x+50}\cdot\dfrac{3x^2+32x+20}{18x^2+27x+10}$

For the following exercises, divide the rational expressions.

24. $\dfrac{3y^2-7y-6}{2y^2-3y-9}\div\dfrac{y^2+y-2}{2y^2+y-3}$

25. $\dfrac{6p^2+p-12}{8p^2+18p+9}\div\dfrac{6p^2-11p+4}{2p^2+11p-6}$

26. $\dfrac{q^2-9}{q^2+6q+9}\div\dfrac{q^2-2q-3}{q^2+2q-3}$

27. $\dfrac{18d^2+77d-18}{27d^2-15d+2}\div\dfrac{3d^2+29d-44}{9d^2-15d+4}$

28. $\dfrac{16x^2+18x-55}{32x^2-36x-11}\div\dfrac{2x^2+17x+30}{4x^2+25x+6}$

29. $\dfrac{144b^2-25}{72b^2-6b-10}\div\dfrac{18b^2-21b+5}{36b^2-18b-10}$

30. $\dfrac{16a^2-24a+9}{4a^2+17a-15}\div\dfrac{16a^2-9}{4a^2+11a+6}$

31. $\dfrac{22y^2+59y+10}{12y^2+28y-5}\div\dfrac{11y^2+46y+8}{24y^2-10y+1}$

32. $\dfrac{9x^2+3x-20}{3x^2-7x+4}\div\dfrac{6x^2+4x-10}{x^2-2x+1}$

For the following exercises, add and subtract the rational expressions, and then simplify.

33. $\frac{4}{x}+\frac{10}{y}$

34. $\frac{12}{2q}-\frac{6}{3p}$

35. $\frac{4}{a+1}+\frac{5}{a-3}$

36. $\frac{c+2}{3}-\frac{c-4}{4}$

37. $\frac{y+3}{y-2}+\frac{y-3}{y+1}$

38. $\frac{x-1}{x+1}-\frac{2x+3}{2x+1}$

39. $\frac{3z}{z+1}+\frac{2z+5}{z-2}$

40. $\frac{4p}{p+1}-\frac{p+1}{4p}$

41. $\frac{x}{x+1}+\frac{y}{y+1}$

For the following exercises, simplify the rational expression.

42. $\dfrac{\frac{6}{y}-\frac{4}{x}}{y}$

43. $\dfrac{\frac{2}{a}+\frac{7}{b}}{b}$

44. $\dfrac{\frac{x}{4}-\frac{p}{8}}{p}$

45. $\dfrac{\frac{3}{a}+\frac{b}{6}}{\frac{2b}{3a}}$

46. $\dfrac{\frac{3}{x+1}+\frac{2}{x-1}}{\frac{x-1}{x+1}}$

47. $\dfrac{\frac{a}{b}-\frac{b}{a}}{\frac{a+b}{ab}}$

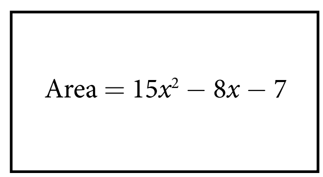
48. $\dfrac{\frac{2x}{3}+\frac{4x}{7}}{\frac{x}{2}}$

49. $\dfrac{\frac{2c}{c+2}+\frac{c-1}{c+1}}{\frac{2c+1}{c+1}}$

50. $\dfrac{\frac{x}{y}-\frac{y}{x}}{\frac{x}{y}+\frac{y}{x}}$

REAL-WORLD APPLICATIONS

51. Brenda is placing tile on her bathroom fl or. The area of the fl or is $15x^2-8x-7$ ft². The area of one tile is x^2-2x+1 ft². To fi d the number of tiles needed, simplify the rational expression: $\frac{15x^2-8x-7}{x^2-2x+1}$.



52. The area of Sandy's yard is $25x^2-625$ ft². A patch of sod has an area of $x^2-10x+25$ ft². Divide the two areas and simplify to fi d how many pieces of sod Sandy needs to cover her yard.

53. Aaron wants to mulch his garden. His garden is $x^2+18x+81$ ft². One bag of mulch covers x^2-81 ft². Divide the expressions and simplify to fi d how many bags of mulch Aaron needs to mulch his garden.

EXTENSIONS

For the following exercises, perform the given operations and simplify.

54. $\frac{x^2+x-6}{x^2-2x-3}\cdot\frac{2x^2-3x-9}{x^2-x-2}\div\frac{10x^2+27x+18}{x^2+2x+1}$

55. $\dfrac{\frac{3y^2-10y+3}{3y^2+5y-2}\cdot\frac{2y^2-3y-20}{2y^2-y-15}}{y-4}$

56. $\dfrac{\frac{4a+1}{2a-3}+\frac{2a-3}{2a+3}}{\frac{4a^2+9}{a}}$

57. $\frac{x^2+7x+12}{x^2+x-6}\div\frac{3x^2+19x+28}{8x^2-4x-24}\div\frac{2x^2+x-3}{3x^2+4x-7}$