LEARNING OBJECTIVES

In this section students will:

- Classify a real number as a natural, whole, integer, rational, or irrational number.
- Perform calculations using order of operations.
- Use the following properties of real numbers: commutative, associative, distributive, inverse, and identity.
- Evaluate algebraic expressions.
- Simplify algebraic expressions.

1.1 REAL NUMBERS: ALGEBRA ESSENTIALS

It is often said that mathematics is the language of science. If this is true, then an essential part of the language of mathematics is numbers. The earliest use of numbers occurred 100 centuries ago in the Middle East to count, or enumerate items. Farmers, cattlemen, and tradesmen used tokens, stones, or markers to signify a single quantity—a sheaf of grain, a head of livestock, or a fixed length of cloth, for example. Doing so made commerce possible, leading to improved communications and the spread of civilization.

Three to four thousand years ago, Egyptians introduced fractions. They first used them to show reciprocals. Later, they used them to represent the amount when a quantity was divided into equal parts.

But what if there were no cattle to trade or an entire crop of grain was lost in a flood? How could someone indicate the existence of nothing? From earliest times, people had thought of a "base state" while counting and used various symbols to represent this null condition. However, it was not until about the fifth century A.D. in India that zero was added to the number system and used as a numeral in calculations.

Clearly, there was also a need for numbers to represent loss or debt. In India, in the seventh century A.D., negative numbers were used as solutions to mathematical equations and commercial debts. The opposites of the counting numbers expanded the number system even further.

Because of the evolution of the number system, we can now perform complex calculations using these and other categories of real numbers. In this section, we will explore sets of numbers, calculations with different kinds of numbers, and the use of numbers in expressions.

Classifying a Real Number

The numbers we use for counting, or enumerating items, are the **natural numbers**: 1, 2, 3, 4, 5, and so on. We describe them in set notation as $\{1, 2, 3, \ldots\}$ where the ellipsis (…) indicates that the numbers continue to infinity. The natural numbers are, of course, also called the *counting numbers*. Any time we enumerate the members of a team, count the coins in a collection, or tally the trees in a grove, we are using the set of natural numbers. The set of **whole numbers** is the set of natural numbers plus zero: $\{0, 1, 2, 3, \ldots\}$.

The set of **integers** adds the opposites of the natural numbers to the set of whole numbers: $\{\ldots, -3, -2, -1, 0, 1, 2, 3, \ldots\}$. It is useful to note that the set of integers is made up of three distinct subsets: negative integers, zero, and positive integers. In this sense, the positive integers are just the natural numbers. Another way to think about it is that the natural numbers are a subset of the integers.

negative integers	zero	positive integers
$\ldots, -3, -2, -1,$	$0,$	$1, 2, 3, \ldots$

The set of **rational numbers** is written as $\left\{ \dfrac{m}{n} \,\middle|\, m \text{ and } n \text{ are integers and } n \neq 0 \right\}$. Notice from the definition that rational numbers are fractions (or quotients) containing integers in both the numerator and the denominator, and the denominator is never 0. We can also see that every natural number, whole number, and integer is a rational number with a denominator of 1.

Because they are fractions, any rational number can also be expressed in decimal form. Any rational number can be represented as either:

1. a terminating decimal: $\dfrac{15}{8} = 1.875$, or
2. a repeating decimal: $\dfrac{4}{11} = 0.36363636\ldots = 0.\overline{36}$

We use a line drawn over the repeating block of numbers instead of writing the group multiple times.

Example 1 Writing Integers as Rational Numbers

Write each of the following as a rational number.

a. 7 **b.** 0 **c.** -8

Solution Write a fraction with the integer in the numerator and 1 in the denominator.

a. $7 = \frac{7}{1}$ **b.** $0 = \frac{0}{1}$ **c.** $-8 = -\frac{8}{1}$

Try It #1

Write each of the following as a rational number.

a. 11 **b.** 3 **c.** -4

Example 2 Identifying Rational Numbers

Write each of the following rational numbers as either a terminating or repeating decimal.

a. $-\frac{5}{7}$ **b.** $\frac{15}{5}$ **c.** $\frac{13}{25}$

Solution Write each fraction as a decimal by dividing the numerator by the denominator.

a. $-\frac{5}{7} = -0.\overline{714285}$, a repeating decimal

b. $\frac{15}{5} = 3$ (or 3.0), a terminating decimal

c. $\frac{13}{25} = 0.52$, a terminating decimal

Try It #2

Write each of the following rational numbers as either a terminating or repeating decimal.

a. $\frac{68}{17}$ **b.** $\frac{8}{13}$ **c.** $-\frac{17}{20}$

Irrational Numbers

At some point in the ancient past, someone discovered that not all numbers are rational numbers. A builder, for instance, may have found that the diagonal of a square with unit sides was not 2 or even $\frac{3}{2}$, but was something else. Or a garment maker might have observed that the ratio of the circumference to the diameter of a roll of cloth was a little bit more than 3, but still not a rational number. Such numbers are said to be *irrational* because they cannot be written as fractions. These numbers make up the set of **irrational numbers**. Irrational numbers cannot be expressed as a fraction of two integers. It is impossible to describe this set of numbers by a single rule except to say that a number is irrational if it is not rational. So we write this as shown.

$$\{h \mid h \text{ is not a rational number}\}$$

Example 3 Differentiating Rational and Irrational Numbers

Determine whether each of the following numbers is rational or irrational. If it is rational, determine whether it is a terminating or repeating decimal.

a. $\sqrt{25}$ **b.** $\frac{33}{9}$ c. $\sqrt{11}$ **d.** $\frac{17}{34}$ e. 0.3033033303333…

Solution

a. $\sqrt{25}$: Th s can be simplifi d as $\sqrt{25} = 5$. Therefore, $\sqrt{25}$ is rational.

b. $\frac{33}{9}$: Because it is a fraction, $\frac{33}{9}$ is a rational number. Next, simplify and divide.

$$\frac{33}{9} = \frac{\overset{11}{\cancel{33}}}{\underset{3}{\cancel{9}}} = \frac{11}{3} = 3.\overline{6}$$

So, $\frac{33}{9}$ is rational and a repeating decimal.

c. $\sqrt{11}$: Th s cannot be simplifi d any further. Therefore, $\sqrt{11}$ is an irrational number.

d. $\frac{17}{34}$: Because it is a fraction, $\frac{17}{34}$ is a rational number. Simplify and divide.

$$\frac{17}{34} = \frac{\overset{1}{\cancel{17}}}{\underset{2}{\cancel{34}}} = \frac{1}{2} = 0.5$$

So, $\frac{17}{34}$ is rational and a terminating decimal.

e. 0.3033033303333 … is not a terminating decimal. Also note that there is no repeating pattern because the group of 3s increases each time. Therefore it is neither a terminating nor a repeating decimal and, hence, not a rational number. It is an irrational number.

Try It #3

Determine whether each of the following numbers is rational or irrational. If it is rational, determine whether it is a terminating or repeating decimal.

a. $\frac{7}{77}$ **b.** $\sqrt{81}$ **c.** 4.27027002700027 … **d.** $\frac{91}{13}$ **e.** $\sqrt{39}$

Real Numbers

Given any number n, we know that n is either rational or irrational. It cannot be both. The sets of rational and irrational numbers together make up the set of **real numbers**. As we saw with integers, the real numbers can be divided into three subsets: negative real numbers, zero, and positive real numbers. Each subset includes fractions, decimals, and irrational numbers according to their algebraic sign (+ or −). Zero is considered neither positive nor negative.

The real numbers can be visualized on a horizontal number line with an arbitrary point chosen as 0, with negative numbers to the left of 0 and positive numbers to the right of 0. A fixed unit distance is then used to mark off each integer (or other basic value) on either side of 0. Any real number corresponds to a unique position on the number line. The converse is also true: Each location on the number line corresponds to exactly one real number. This is known as a one-to-one correspondence. We refer to this as the **real number line** as shown in **Figure 2**.

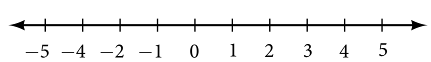

Figure 1 The real number line

Example 4 **Classifying Real Numbers**

Classify each number as either positive or negative and as either rational or irrational. Does the number lie to the left or the right of 0 on the number line?

a. $-\frac{10}{3}$ **b.** $\sqrt{5}$ **c.** $-\sqrt{289}$ **d.** -6π **e.** 0.615384615384 …

Solution

a. $-\frac{10}{3}$ is negative and rational. It lies to the left f 0 on the number line.

b. $\sqrt{5}$ is positive and irrational. It lies to the right of 0.

c. $-\sqrt{289} = -\sqrt{17^2} = -17$ is negative and rational. It lies to the left f 0.

d. -6π is negative and irrational. It lies to the left f 0.

e. 0.615384615384 … is a repeating decimal so it is rational and positive. It lies to the right of 0.

Try It #4

Classify each number as either positive or negative and as either rational or irrational. Does the number lie to the left or the right of 0 on the number line?

a. $\sqrt{73}$ **b.** $-11.411411411\ldots$ **c.** $\frac{47}{19}$ **d.** $-\frac{\sqrt{5}}{2}$ **e.** 6.210735

Sets of Numbers as Subsets

Beginning with the natural numbers, we have expanded each set to form a larger set, meaning that there is a subset relationship between the sets of numbers we have encountered so far. These relationships become more obvious when seen as a diagram, such as **Figure 3**.

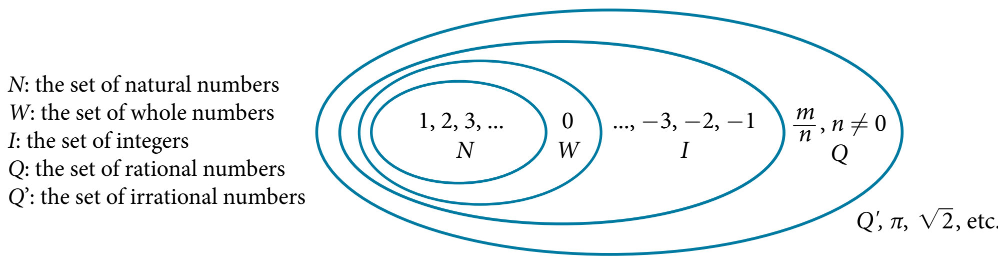

Figure 2 Sets of numbers

sets of numbers

The set of **natural numbers** includes the numbers used for counting: {1, 2, 3, ...}.

The set of **whole numbers** is the set of natural numbers plus zero: {0, 1, 2, 3, ...}.

The set of **integers** adds the negative natural numbers to the set of whole numbers: {..., −3, −2, −1, 0, 1, 2, 3, ... }.

The set of **rational numbers** includes fractions written as $\left\{ \frac{m}{n} \middle| m \text{ and n are integers and } n \neq 0 \right\}$

The set of **irrational numbers** is the set of numbers that are not rational, are nonrepeating, and are nonterminating: $\{h \mid h$ is not a rational number$\}$.

Example 5 Differentiating the Sets of Numbers

Classify each number as being a natural number (N), whole number (W), integer (I), rational number (Q), and/or irrational number (Q').

a. $\sqrt{36}$ **b.** $\frac{8}{3}$ **c.** $\sqrt{73}$ **d.** -6 **e.** $3.2121121112\ldots$

Solution

	N	W	I	Q	Q'
a. $\sqrt{36} = 6$	×	×	×	×	
b. $\frac{8}{3} = 2.\overline{6}$				×	
c. $\sqrt{73}$					×
d. -6			×	×	
e. 3.2121121112...					×

Try It #5

Classify each number as being a natural number (N), whole number (W), integer (I), rational number (Q), and/or irrational number (Q').

a. $-\frac{35}{7}$ **b.** 0 **c.** $\sqrt{169}$ **d.** $\sqrt{24}$ **e.** $4.763763763\ldots$

Performing Calculations Using the Order of Operations

When we multiply a number by itself, we square it or raise it to a power of 2. For example, $4^2 = 4 \cdot 4 = 16$. We can raise any number to any power. In general, the **exponential notation** a^n means that the number or variable a is used as a factor n times.

$$a^n = \overbrace{a \cdot a \cdot a \cdot \ldots \cdot a}^{n \text{ factors}}$$

In this notation, a^n is read as the nth power of a, where a is called the **base** and n is called the **exponent**. A term in exponential notation may be part of a mathematical expression, which is a combination of numbers and operations. For example, $24 + 6 \cdot \frac{2}{3} - 4^2$ is a mathematical expression.

To evaluate a mathematical expression, we perform the various operations. However, we do not perform them in any random order. We use the **order of operations**. This is a sequence of rules for evaluating such expressions.

Recall that in mathematics we use parentheses ( ), brackets [ ], and braces { } to group numbers and expressions so that anything appearing within the symbols is treated as a unit. Additionally, fraction bars, radicals, and absolute value bars are treated as grouping symbols. When evaluating a mathematical expression, begin by simplifying expressions within grouping symbols.

The next step is to address any exponents or radicals. Afterward, perform multiplication and division from left to right and finally addition and subtraction from left o right.

Let's take a look at the expression provided.

$$24 + 6 \cdot \frac{2}{3} - 4^2$$

There are no grouping symbols, so we move on to exponents or radicals. The number 4 is raised to a power of 2, so simplify 4^2 as 16.

$$24 + 6 \cdot \frac{2}{3} - 4^2$$

$$24 + 6 \cdot \frac{2}{3} - 16$$

Next, perform multiplication or division, left o right.

$$24 + 6 \cdot \frac{2}{3} - 16$$

$$24 + 4 - 16$$

Lastly, perform addition or subtraction, left o right.

$$24 + 4 - 16$$
$$28 - 16$$
$$12$$

Therefore, $24 + 6 \cdot \frac{2}{3} - 4^2 = 12$.

For some complicated expressions, several passes through the order of operations will be needed. For instance, there may be a radical expression inside parentheses that must be simplified before the parentheses are evaluated. Following the order of operations ensures that anyone simplifying the same mathematical expression will get the same result.

order of operations

Operations in mathematical expressions must be evaluated in a systematic order, which can be simplified using the acronym **PEMDAS**:

P(arentheses)
E(xponents)
M(ultiplication) and **D**(ivision)
A(ddition) and **S**(ubtraction)

How To…

Given a mathematical expression, simplify it using the order of operations.

1. Simplify any expressions within grouping symbols.
2. Simplify any expressions containing exponents or radicals.
3. Perform any multiplication and division in order, from left o right.
4. Perform any addition and subtraction in order, from left o right.

Example 6 Using the Order of Operations

Use the order of operations to evaluate each of the following expressions.

a. $(3 \cdot 2)^2 - 4(6 + 2)$ **b.** $\dfrac{5^2 - 4}{7} - \sqrt{11 - 2}$ **c.** $6 - |5 - 8| + 3(4 - 1)$

d. $\dfrac{14 - 3 \cdot 2}{2 \cdot 5 - 3^2}$ **e.** $7(5 \cdot 3) - 2[(6 - 3) - 4^2] + 1$

Solution

a.

$$\begin{aligned} (3 \cdot 2)^2 - 4(6 + 2) &= (6)^2 - 4(8) && \text{Simplify parentheses.} \\ &= 36 - 4(8) && \text{Simplify exponent.} \\ &= 36 - 32 && \text{Simplify multiplication.} \\ &= 4 && \text{Simplify subtraction.} \end{aligned}$$

b.

$$\begin{aligned} \frac{5^2 - 4}{7} - \sqrt{11 - 2} &= \frac{5^2 - 4}{7} - \sqrt{9} && \text{Simplify grouping symbols (radical).} \\ &= \frac{5^2 - 4}{7} - 3 && \text{Simplify radical.} \\ &= \frac{25 - 4}{7} - 3 && \text{Simplify exponent.} \\ &= \frac{21}{7} - 3 && \text{Simplify subtraction in numerator.} \\ &= 3 - 3 && \text{Simplify division.} \\ &= 0 && \text{Simplify subtraction.} \end{aligned}$$

Note that in the first step, the radical is treated as a grouping symbol, like parentheses. Also, in the third step, the fraction bar is considered a grouping symbol so the numerator is considered to be grouped.

c.

$$\begin{aligned} 6 - |5 - 8| + 3(4 - 1) &= 6 - |-3| + 3(3) && \text{Simplify inside grouping symbols.} \\ &= 6 - 3 + 3(3) && \text{Simplify absolute value.} \\ &= 6 - 3 + 9 && \text{Simplify multiplication.} \\ &= 3 + 9 && \text{Simplify subtraction.} \\ &= 12 && \text{Simplify addition.} \end{aligned}$$

d.

$$\begin{aligned} \frac{14 - 3 \cdot 2}{2 \cdot 5 - 3^2} &= \frac{14 - 3 \cdot 2}{2 \cdot 5 - 9} && \text{Simplify exponent.} \\ &= \frac{14 - 6}{10 - 9} && \text{Simplify products.} \\ &= \frac{8}{1} && \text{Simplify diffe ences.} \\ &= 8 && \text{Simplify quotient.} \end{aligned}$$

In this example, the fraction bar separates the numerator and denominator, which we simplify separately until the last step.

e

$$\begin{aligned} 7(5 \cdot 3) - 2[(6 - 3) - 4^2] + 1 &= 7(15) - 2[(3) - 4^2] + 1 && \text{Simplify inside parentheses.} \\ &= 7(15) - 2(3 - 16) + 1 && \text{Simplify exponent.} \\ &= 7(15) - 2(-13) + 1 && \text{Subtract.} \\ &= 105 + 26 + 1 && \text{Multiply.} \\ &= 132 && \text{Add.} \end{aligned}$$

Try It #6

Use the order of operations to evaluate each of the following expressions.

a. $\sqrt{5^2 - 4^2} + 7(5 - 4)^2$ **b.** $1 + \dfrac{7 \cdot 5 - 8 \cdot 4}{9 - 6}$ **c.** $|1.8 - 4.3| + 0.4\sqrt{15 + 10}$

d. $\dfrac{1}{2}[5 \cdot 3^2 - 7^2] + \dfrac{1}{3} \cdot 9^2$ **e.** $[(3 - 8)^2 - 4] - (3 - 8)$

Using Properties of Real Numbers

For some activities we perform, the order of certain operations does not matter, but the order of other operations does. For example, it does not make a difference if we put on the right shoe before the left or vice-versa. However, it does matter whether we put on shoes or socks first. The same thing is true for operations in mathematics.

Commutative Properties

Th **commutative property of addition** states that numbers may be added in any order without affecting the sum.

$$a + b = b + a$$

We can better see this relationship when using real numbers.

$$(-2) + 7 = 5 \qquad \text{and} \qquad 7 + (-2) = 5$$

Similarly, the **commutative property of multiplication** states that numbers may be multiplied in any order without affecting the product.

$$a \cdot b = b \cdot a$$

Again, consider an example with real numbers.

$$(-11) \cdot (-4) = 44 \qquad \text{and} \qquad (-4) \cdot (-11) = 44$$

It is important to note that neither subtraction nor division is commutative. For example, $17 - 5$ is not the same as $5 - 17$. Similarly, $20 \div 5 \neq 5 \div 20$.

Associative Properties

Th **associative property of multiplication** tells us that it does not matter how we group numbers when multiplying. We can move the grouping symbols to make the calculation easier, and the product remains the same.

$$a(bc) = (ab)c$$

Consider this example.

$$(3 \cdot 4) \cdot 5 = 60 \qquad \text{and} \qquad 3 \cdot (4 \cdot 5) = 60$$

Th **associative property of addition** tells us that numbers may be grouped differently without affecting the sum.

$$a + (b + c) = (a + b) + c$$

This property can be especially helpful when dealing with negative integers. Consider this example.

$$[15 + (-9)] + 23 = 29 \qquad \text{and} \qquad 15 + [(-9) + 23] = 29$$

Are subtraction and division associative? Review these examples.

$$\begin{aligned} 8 - (3 - 15) &\stackrel{?}{=} (8 - 3) - 15 \\ 8 - (-12) &\stackrel{?}{=} 5 - 15 \\ 20 &\neq -10 \end{aligned} \qquad\qquad \begin{aligned} 64 \div (8 \div 4) &\stackrel{?}{=} (64 \div 8) \div 4 \\ 64 \div 2 &\stackrel{?}{=} 8 \div 4 \\ 32 &\neq 2 \end{aligned}$$

As we can see, neither subtraction nor division is associative.

Distributive Property

Th **distributive property** states that the product of a factor times a sum is the sum of the factor times each term in the sum.

$$a \cdot (b + c) = a \cdot b + a \cdot c$$

This property combines both addition and multiplication (and is the only property to do so). Let us consider an example.

$$\begin{aligned} 4 \cdot [12 + (-7)] &= \overbrace{4 \cdot 12} + \overbrace{4 \cdot (-7)} \\ &= 48 + (-28) \\ &= 20 \end{aligned}$$

Note that 4 is outside the grouping symbols, so we distribute the 4 by multiplying it by 12, multiplying it by −7, and adding the products.

To be more precise when describing this property, we say that multiplication distributes over addition. The reverse is not true, as we can see in this example.

$$\begin{aligned} 6 + (3 \cdot 5) &\stackrel{?}{=} (6 + 3) \cdot (6 + 5) \\ 6 + (15) &\stackrel{?}{=} (9) \cdot (11) \\ 21 &\neq 99 \end{aligned}$$

A special case of the distributive property occurs when a sum of terms is subtracted.

$$a - b = a + (-b)$$

For example, consider the difference $12 - (5 + 3)$. We can rewrite the difference of the two terms 12 and $(5 + 3)$ by turning the subtraction expression into addition of the opposite. So instead of subtracting $(5 + 3)$, we add the opposite.

$$12 + (-1) \cdot (5 + 3)$$

Now, distribute −1 and simplify the result.

$$\begin{aligned} 12 - (5 + 3) &= 12 + (-1) \cdot (5 + 3) \\ &= 12 + [(-1) \cdot 5 + (-1) \cdot 3] \\ &= 12 + (-8) \\ &= 4 \end{aligned}$$

This seems like a lot of trouble for a simple sum, but it illustrates a powerful result that will be useful once we introduce algebraic terms. To subtract a sum of terms, change the sign of each term and add the results. With this in mind, we can rewrite the last example.

$$\begin{aligned} 12 - (5 + 3) &= 12 + (-5 - 3) \\ &= 12 + (-8) \\ &= 4 \end{aligned}$$

Identity Properties

The **identity property of addition** states that there is a unique number, called the additive identity (0) that, when added to a number, results in the original number.

$$a + 0 = a$$

The **identity property of multiplication** states that there is a unique number, called the multiplicative identity (1) that, when multiplied by a number, results in the original number.

$$a \cdot 1 = a$$

For example, we have $(-6) + 0 = -6$ and $23 \cdot 1 = 23$. There are no exceptions for these properties; they work for every real number, including 0 and 1.

Inverse Properties

Th **inverse property of addition** states that, for every real number a, there is a unique number, called the additive inverse (or opposite), denoted$-a$, that, when added to the original number, results in the additive identity, 0.

$$a + (-a) = 0$$

For example, if $a = -8$, the additive inverse is 8, since $(-8) + 8 = 0$.

Th **inverse property of multiplication** holds for all real numbers except 0 because the reciprocal of 0 is not defined. The property states that, for every real number a, there is a unique number, called the multiplicative inverse (or reciprocal), denoted $\frac{1}{a}$, that, when multiplied by the original number, results in the multiplicative identity, 1.

$$a \cdot \frac{1}{a} = 1$$

For example, if $a = -\frac{2}{3}$, the reciprocal, denoted $\frac{1}{a}$, is $-\frac{3}{2}$ because

$$a \cdot \frac{1}{a} = \left(-\frac{2}{3}\right) \cdot \left(-\frac{3}{2}\right) = 1$$

properties of real numbers

The following properties hold for real numbers a, b, and c.

	Addition	**Multiplication**
Commutative Property	$a + b = b + a$	$a \cdot b = b \cdot a$
Associative Property	$a + (b + c) = (a + b) + c$	$a(bc) = (ab)c$
Distributive Property	$a \cdot (b + c) = a \cdot b + a \cdot c$	
Identity Property	There exists a unique real number called the additive identity, 0, such that, for any real number a $a + 0 = a$	There exists a unique real number called the multiplicative identity, 1, such that, for any real number a $a \cdot 1 = a$
Inverse Property	Every real number a has an additive inverse, or opposite, denoted $-a$, such that $a + (-a) = 0$	Every nonzero real number a has a multiplicative inverse, or reciprocal, denoted $\frac{1}{a}$, such that $a \cdot \left(\frac{1}{a}\right) = 1$

Example 7 Using Properties of Real Numbers

Use the properties of real numbers to rewrite and simplify each expression. State which properties apply.

a. $3 \cdot 6 + 3 \cdot 4$ **b.** $(5 + 8) + (-8)$ **c.** $6 - (15 + 9)$ **d.** $\frac{4}{7} \cdot \left(\frac{2}{3} \cdot \frac{7}{4}\right)$ **e.** $100 \cdot [0.75 + (-2.38)]$

Solution

a.
$$\begin{aligned} 3 \cdot 6 + 3 \cdot 4 &= 3 \cdot (6 + 4) && \text{Distributive property} \\ &= 3 \cdot 10 && \text{Simplify.} \\ &= 30 && \text{Simplify.} \end{aligned}$$

b.
$$\begin{aligned} (5 + 8) + (-8) &= 5 + [8 + (-8)] && \text{Associative property of addition} \\ &= 5 + 0 && \text{Inverse property of addition} \\ &= 5 && \text{Identity property of addition} \end{aligned}$$

c.
$$\begin{aligned} 6 - (15 + 9) &= 6 + [(-15) + (-9)] && \text{Distributive property} \\ &= 6 + (-24) && \text{Simplify.} \\ &= -18 && \text{Simplify.} \end{aligned}$$

d.
$$\begin{aligned} \frac{4}{7} \cdot \left(\frac{2}{3} \cdot \frac{7}{4}\right) &= \frac{4}{7} \cdot \left(\frac{7}{4} \cdot \frac{2}{3}\right) && \text{Commutative property of multiplication} \\ &= \left(\frac{4}{7} \cdot \frac{7}{4}\right) \cdot \frac{2}{3} && \text{Associative property of multiplication} \\ &= 1 \cdot \frac{2}{3} && \text{Inverse property of multiplication} \\ &= \frac{2}{3} && \text{Identity property of multiplication} \end{aligned}$$

e.
$$\begin{aligned} 100 \cdot [0.75 + (-2.38)] &= 100 \cdot 0.75 + 100 \cdot (-2.38) && \text{Distributive property} \\ &= 75 + (-238) && \text{Simplify.} \\ &= -163 && \text{Simplify.} \end{aligned}$$

Try It #7

Use the properties of real numbers to rewrite and simplify each expression. State which properties apply.

a. $\left(-\frac{23}{5}\right) \cdot \left[11 \cdot \left(-\frac{5}{23}\right)\right]$ **b.** $5 \cdot (6.2 + 0.4)$ **c.** $18 - (7 - 15)$ **d.** $\frac{17}{18} + \left[\frac{4}{9} + \left(-\frac{17}{18}\right)\right]$ **e.** $6 \cdot (-3) + 6 \cdot 3$

Evaluating Algebraic Expressions

So far, the mathematical expressions we have seen have involved real numbers only. In mathematics, we may see expressions such as $x + 5$, $\frac{4}{3}\pi r^3$, or $\sqrt{2m^3n^2}$. In the expression $x + 5$, 5 is called a **constant** because it does not vary and x is called a **variable** because it does. (In naming the variable, ignore any exponents or radicals containing the variable.) An **algebraic expression** is a collection of constants and variables joined together by the algebraic operations of addition, subtraction, multiplication, and division.

We have already seen some real number examples of exponential notation, a shorthand method of writing products of the same factor. When variables are used, the constants and variables are treated the same way.

$$(-3)^5 = (-3)\cdot(-3)\cdot(-3)\cdot(-3)\cdot(-3) \qquad x^5 = x\cdot x\cdot x\cdot x\cdot x$$

$$(2\cdot 7)^3 = (2\cdot 7)\cdot(2\cdot 7)\cdot(2\cdot 7) \qquad (yz)^3 = (yz)\cdot(yz)\cdot(yz)$$

In each case, the exponent tells us how many factors of the base to use, whether the base consists of constants or variables.

Any variable in an algebraic expression may take on or be assigned different values. When that happens, the value of the algebraic expression changes. To evaluate an algebraic expression means to determine the value of the expression for a given value of each variable in the expression. Replace each variable in the expression with the given value, then simplify the resulting expression using the order of operations. If the algebraic expression contains more than one variable, replace each variable with its assigned value and simplify the expression as before.

Example 8 Describing Algebraic Expressions

List the constants and variables for each algebraic expression.

a. $x + 5$ **b.** $\frac{4}{3}\pi r^3$ **c.** $\sqrt{2m^3n^2}$

Solution

	Constants	Variables
a. $x + 5$	5	x
b. $\frac{4}{3}\pi r^3$	$\frac{4}{3}, \pi$	r
c. $\sqrt{2m^3n^2}$	2	m, n

Try It #8

List the constants and variables for each algebraic expression.

a. $2\pi r(r + h)$ **b.** $2(L + W)$ **c.** $4y^3 + y$

Example 9 Evaluating an Algebraic Expression at Different Values

Evaluate the expression $2x - 7$ for each value for x.

a. $x = 0$ **b.** $x = 1$ **c.** $x = \frac{1}{2}$ **d.** $x = -4$

Solution

a. Substitute 0 for x.

$$\begin{aligned} 2x - 7 &= 2(0) - 7 \\ &= 0 - 7 \\ &= -7 \end{aligned}$$

b. Substitute 1 for x.

$$\begin{aligned} 2x - 7 &= 2(1) - 7 \\ &= 2 - 7 \\ &= -5 \end{aligned}$$

c. Substitute $\frac{1}{2}$ for x.

$$\begin{aligned} 2x - 7 &= 2\left(\frac{1}{2}\right) - 7 \\ &= 1 - 7 \\ &= -6 \end{aligned}$$

d. Substitute -4 for x.

$$\begin{aligned} 2x - 7 &= 2(-4) - 7 \\ &= -8 - 7 \\ &= -15 \end{aligned}$$

Try It #9

Evaluate the expression $11 - 3y$ for each value for y.

a. $y = 2$ **b.** $y = 0$ **c.** $y = \frac{2}{3}$ **d.** $y = -5$

Example 10 Evaluating Algebraic Expressions

Evaluate each expression for the given values.

a. $x + 5$ for $x = -5$ **b.** $\frac{t}{2t-1}$ for $t = 10$ **c.** $\frac{4}{3}\pi r^3$ for $r = 5$

d. $a + ab + b$ for $a = 11$, $b = -8$ **e.** $\sqrt{2m^3n^2}$ for $m = 2$, $n = 3$

Solution

a. Substitute -5 for x.

$$\begin{aligned} x + 5 &= (-5) + 5 \\ &= 0 \end{aligned}$$

b. Substitute 10 for t.

$$\begin{aligned} \frac{t}{2t - 1} &= \frac{(10)}{2(10) - 1} \\ &= \frac{10}{20 - 1} \\ &= \frac{10}{19} \end{aligned}$$

c. Substitute 5 for r.

$$\begin{aligned} \frac{4}{3}\pi r^3 &= \frac{4}{3}\pi(5)^3 \\ &= \frac{4}{3}\pi(125) \\ &= \frac{500}{3}\pi \end{aligned}$$

d. Substitute 11 for a and -8 for b.

$$\begin{aligned} a + ab + b &= (11) + (11)(-8) + (-8) \\ &= 11 - 88 - 8 \\ &= -85 \end{aligned}$$

e. Substitute 2 for m and 3 for n.

$$\begin{aligned} \sqrt{2m^3n^2} &= \sqrt{2(2)^3(3)^2} \\ &= \sqrt{2(8)(9)} \\ &= \sqrt{144} \\ &= 12 \end{aligned}$$

Try It #10

Evaluate each expression for the given values.

a. $\frac{y + 3}{y - 3}$ for $y = 5$ **b.** $7 - 2t$ for $t = -2$ **c.** $\frac{1}{3}\pi r^2$ for $r = 11$

d. $(p^2q)^3$ for $p = -2$, $q = 3$ **e.** $4(m - n) - 5(n - m)$ for $m = \frac{2}{3}$, $n = \frac{1}{3}$

Formulas

An **equation** is a mathematical statement indicating that two expressions are equal. The expressions can be numerical or algebraic. The equation is not inherently true or false, but only a proposition. The values that make the equation true, the solutions, are found using the properties of real numbers and other results. For example, the equation $2x + 1 = 7$ has the unique solution $x = 3$ because when we substitute 3 for x in the equation, we obtain the true statement $2(3) + 1 = 7$.

A **formula** is an equation expressing a relationship between constant and variable quantities. Very often, the equation is a means of finding the value of one quantity (often a single variable) in terms of another or other quantities. One of the most common examples is the formula for finding the area A of a circle in terms of the radius r of the circle: $A = \pi r^2$. For any value of r, the area A can be found by evaluating the expression πr^2.

Example 11 **Using a Formula**

A right circular cylinder with radius r and height h has the surface area S (in square units) given by the formula $S = 2\pi r(r + h)$. See **Figure 4**. Find the surface area of a cylinder with radius 6 in. and height 9 in. Leave the answer in terms of π.

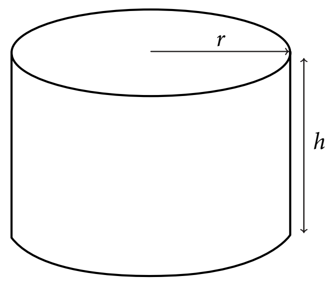

Figure 3 Right circular cylinder

Solution Evaluate the expression $2\pi r(r + h)$ for $r = 6$ and $h = 9$.

$$\begin{aligned} S &= 2\pi r(\mathrm{r} + \mathrm{h}) \\ &= 2\pi(6)[(6) + (9)] \\ &= 2\pi(6)(15) \\ &= 180\pi \end{aligned}$$

The surface area is 180π square inches.

Try It #11

A photograph with length L and width W is placed in a matte of width 8 centimeters (cm). The area of the matte (in square centimeters, or cm^2) is found to be $A = (L + 16)(W + 16) - L \cdot \text{W}$. See **Figure 5**. Find the area of a matte for a photograph with length 32 cm and width 24 cm.

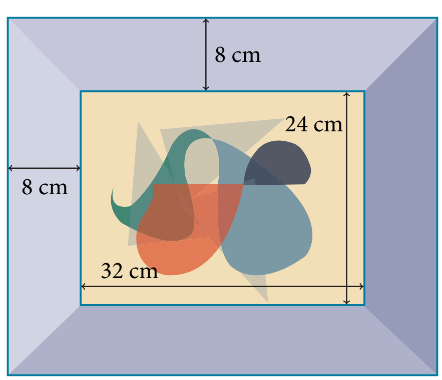

Figure 4

Simplifying Algebraic Expressions

Sometimes we can simplify an algebraic expression to make it easier to evaluate or to use in some other way. To do so, we use the properties of real numbers. We can use the same properties in formulas because they contain algebraic expressions.

Example 12 Simplifying Algebraic Expressions

Simplify each algebraic expression.

a. $3x - 2y + x - 3y - 7$ **b.** $2r - 5(3 - r) + 4$ **c.** $\left(4t - \frac{5}{4}s\right) - \left(\frac{2}{3}t + 2s\right)$ **d.** $2mn - 5m + 3mn + n$

Solution

a.

$$\begin{aligned} 3x - 2y + x - 3y - 7 &= 3x + x - 2y - 3y - 7 && \text{Commutative property of addition} \\ &= 4x - 5y - 7 && \text{Simplify.} \end{aligned}$$

b.

$$\begin{aligned} 2r - 5(3 - r) + 4 &= 2r - 15 + 5r + 4 && \text{Distributive property} \\ &= 2r + 5r - 15 + 4 && \text{Commutative property of addition} \\ &= 7r - 11 && \text{Simplify.} \end{aligned}$$

c.

$$\begin{aligned} 4t - 4\left(t - \frac{5}{4}s\right) - \left(\frac{2}{3}t + 2s\right) &= 4t - \frac{5}{4}s - \frac{2}{3}t - 2s && \text{Distributive property} \\ &= 4t - \frac{2}{3}t - \frac{5}{4}s - 2s && \text{Commutative property of addition} \\ &= \frac{10}{3}t - \frac{13}{4}s && \text{Simplify.} \end{aligned}$$

d.

$$\begin{aligned} mn - 5m + 3mn + n &= 2mn + 3mn - 5m + n && \text{Commutative property of addition} \\ &= 5mn - 5m + n && \text{Simplify.} \end{aligned}$$

Try It #12

Simplify each algebraic expression.

a. $\frac{2}{3}y - 2\left(\frac{4}{3}y + z\right)$ **b.** $\frac{5}{t} - 2 - \frac{3}{t} + 1$ **c.** $4p(q - 1) + q(1 - p)$ **d.** $9r - (s + 2r) + (6 - s)$

Example 13 Simplifying a Formula

A rectangle with length L and width W has a perimeter P given by $P = L + W + L + W$. Simplify this expression.

Solution

$$\begin{aligned} P &= L + W + L + W && \\ P &= L + L + W + W && \text{Commutative property of addition} \\ P &= 2L + 2W && \text{Simplify.} \\ P &= 2(L + W) && \text{Distributive property} \end{aligned}$$

Try It #13

If the amount P is deposited into an account paying simple interest r for time t, the total value of the deposit A is given by $A = P + Prt$. Simplify the expression. (Th s formula will be explored in more detail later in the course.)

Access these online resources for additional instruction and practice with real numbers.

- Simplify an Expression (http://openstaxcollege.org/l/simexpress)
- Evaluate an Expression1 (http://openstaxcollege.org/l/ordofoper1)
- Evaluate an Expression2 (http://openstaxcollege.org/l/ordofoper2)

1.1 SECTION EXERCISES

VERBAL

1. Is $\sqrt{2}$ an example of a rational terminating, rational repeating, or irrational number? Tell why it fits that category.

2. What is the order of operations? What acronym is used to describe the order of operations, and what does it stand for?

3. What do the Associative Properties allow us to do when following the order of operations? Explain your answer.

NUMERIC

For the following exercises, simplify the given expression.

4. $10 + 2 \cdot (5 - 3)$

5. $6 \div 2 - (81 \div 3^2)$

6. $18 + (6 - 8)^3$

7. $-2 \cdot [16 \div (8 - 4)^2]^2$

8. $4 - 6 + 2 \cdot 7$

9. $3(5 - 8)$

10. $4 + 6 - 10 \div 2$

11. $12 \div (36 \div 9) + 6$

12. $(4 + 5)^2 \div 3$

13. $3 - 12 \cdot 2 + 19$

14. $2 + 8 \cdot 7 \div 4$

15. $5 + (6 + 4) - 11$

16. $9 - 18 \div 3^2$

17. $14 \cdot 3 \div 7 - 6$

18. $9 - (3 + 11) \cdot 2$

19. $6 + 2 \cdot 2 - 1$

20. $64 \div (8 + 4 \cdot 2)$

21. $9 + 4(2^2)$

22. $(12 \div 3 \cdot 3)^2$

23. $25 \div 5^2 - 7$

24. $(15 - 7) \cdot (3 - 7)$

25. $2 \cdot 4 - 9(-1)$

26. $4^2 - 25 \cdot \frac{1}{5}$

27. $12(3 - 1) \div 6$

ALGEBRAIC

For the following exercises, solve for the variable.

28. $8(x + 3) = 64$

29. $4y + 8 = 2y$

30. $(11a + 3) - 18a = -4$

31. $4z - 2z(1 + 4) = 36$

32. $4y(7 - 2)^2 = -200$

33. $-(2x)^2 + 1 = -3$

34. $8(2 + 4) - 15b = b$

35. $2(11c - 4) = 36$

36. $4(3 - 1)x = 4$

37. $\frac{1}{4}(8w - 4^2) = 0$

For the following exercises, simplify the expression.

38. $4x + x(13 - 7)$

39. $2y - (4)^2 y - 11$

40. $\frac{a}{2^3}(64) - 12a \div 6$

41. $8b - 4b(3) + 1$

42. $5l \div 3l \cdot (9 - 6)$

43. $7z - 3 + z \cdot 6^2$

44. $4 \cdot 3 + 18x \div 9 - 12$

45. $9(y + 8) - 27$

46. $\left(\frac{9}{6}t - 4\right)2$

47. $6 + 12b - 3 \cdot 6b$

48. $18y - 2(1 + 7y)$

49. $\left(\frac{4}{9}\right)^2 \cdot 27x$

50. $8(3 - m) + 1(-8)$

51. $9x + 4x(2 + 3) - 4(2x + 3x)$

52. $5^2 - 4(3x)$

REAL-WORLD APPLICATIONS

For the following exercises, consider this scenario: Fred earns \$40 mowing lawns. He spends \$10 on mp3s, puts half of what is left n a savings account, and gets another \$5 for washing his neighbor's car.

53. Write the expression that represents the number of dollars Fred keeps (and does not put in his savings account). Remember the order of operations.

54. How much money does Fred keep?

For the following exercises, solve the given problem.

55. According to the U.S. Mint, the diameter of a quarter is 0.955 inches. The circumference of the quarter would be the diameter multiplied by π. Is the circumference of a quarter a whole number, a rational number, or an irrational number?

56. Jessica and her roommate, Adriana, have decided to share a change jar for joint expenses. Jessica put her loose change in the jar fi st, and then Adriana put her change in the jar. We know that it does not matter in which order the change was added to the jar. What property of addition describes this fact?

For the following exercises, consider this scenario: There is a mound of g pounds of gravel in a quarry. Throughout the day, 400 pounds of gravel is added to the mound. Two orders of 600 pounds are sold and the gravel is removed from the mound. At the end of the day, the mound has 1,200 pounds of gravel.

57. Write the equation that describes the situation.

58. Solve for g.

For the following exercise, solve the given problem.

59. Ramon runs the marketing department at his company. His department gets a budget every year, and every year, he must spend the entire budget without going over. If he spends less than the budget, then his department gets a smaller budget the following year. At the beginning of this year, Ramon got \$2.5 million for the annual marketing budget. He must spend the budget such that $2{,}500{,}000 - x = 0$. What property of addition tells us what the value of x must be?

TECHNOLOGY

For the following exercises, use a graphing calculator to solve for x. Round the answers to the nearest hundredth.

60. $0.5(12.3)^2 - 48x = \frac{3}{5}$

61. $(0.25 - 0.75)^2 x - 7.2 = 9.9$

EXTENSIONS

62. If a whole number is not a natural number, what must the number be?

63. Determine whether the statement is true or false: The multiplicative inverse of a rational number is also rational.

64. Determine whether the statement is true or false: The product of a rational and irrational number is always irrational.

65. Determine whether the simplifi d expression is rational or irrational: $\sqrt{-18 - 4(5)(-1)}$.

66. Determine whether the simplifi d expression is rational or irrational: $\sqrt{-16 + 4(5) + 5}$.

67. The division of two whole numbers will always result in what type of number?

68. What property of real numbers would simplify the following expression: $4 + 7(x - 1)$?