0.1 Applications of Exponential and Log Functions

Growth

If the rate of population change is positive and directly related to the present population, we have exponential growth. (If change is negative, we have exponential decay which we will examine separately.)

$$y = y_0 e^{kt}$$

It is possible–and sometimes practical–to use other exponential bases besides e, yet this equation $y = y_0 e^{kt}$ has the attractive property of being the simplest form of an equation for some common problems.

Annual Compound Interest

"The rich get richer" or "the more you have the more you make" is compound interest explained in plain English. In contrast to compound interest, simple interest only applies to the principle.

A $1000 saving share earning simple interest of 5% per year would earn 5% of $1000 or $50 per year. Simple interest is, well, simple. The $1000 would earn $50 each year, year after year. In 20 years, $1000 at 5% simple interest would earn $1000. Does anyone actually do this? Not really in the sense that the interest would be ignored for 20 years to only end up with $2000.

In reality, interest is reinvested as it is earned thus earning compound interest. If one earns 5% interest per year compounded annually, the initial investment of $1000 would become $1000 + $50 = $1050 after the first year. Then, 5% of $1050 is $52.50, and we would have $1050 + $52.50 = $1102.50 after the seond year. The shortcut formula for this type of problem is

$$A = P(1 + r)^t$$

where P is the principal or initial investment, r is the annual interest rate as a decimal, t the number of years of investment, and A is the amount earned after t years. For our initial investment of $1000 at 5% annual interest compounded 1 time pear year for 20 years, we would get $A = 1000(1 + .05)^{20} = \$2653.29$.

Compound Interest

For compounding of interest more often than once per year, we have

$$A = P\left(1 + \frac{r}{k}\right)^{kt}$$

A = amount after t years
P = principal, the amount invested at time t = 0
t = time elapsed in years
r = annual percentage rate (APR) as a decimal
k = number of compounding periods per year

Example Suppose that we invest $1000 at 5% APR for 20 years, as above, but consider different compounding periods.

quarterly (k=4):

$$\begin{aligned} A &= 1000\left(1 + \frac{.04}{4}\right)^{4\cdot 120} \\ &= 1000(1.01)^{80} \\ &= \$2216.72 \end{aligned}$$

monthly (k=12):

$$\begin{aligned} A &= 1000(\left(1 + \frac{.04}{12}\right)^{12\cdot 20} \\ &= \$2222.58 \end{aligned}$$

daily (k=365):

$$\begin{aligned} A &= 1000\left(1 + \frac{.04}{365}\right)^{365\cdot 20} \\ &= \$2225.44 \end{aligned}$$

hourly (k=365*24=8760):

$$\begin{aligned} A &= 1000\left(1 + \frac{.04}{8760}\right)^{8760\cdot 20} \\ &= \$2225.54 \end{aligned}$$

every minute (k=365*24*60=525600):

$$\begin{aligned} A &= 1000\left(1 + \frac{.04}{525600}\right)^{525600\cdot 20} \\ &= \$2225.54 \end{aligned}$$

There was actually a slight difference between the calculation for hourly and every-minute compounding of interest, but the difference was less than half a cent and did not show up when we rounded to the closest penny.

Continuous Compounding of Interest

If compunding is done continuously, that is infinitely many times per second, then we have

$$A = Pe^{rt}$$

Example Suppose that we invest \$1000 at 5% APR for 20 years, as above, but we compound the interest continuously.

$$A = 1000e^{.04(20)} = \$2225.54$$

This figure is slightly greater than per-minute compounding, but very close.

Example How long will it take \$1000 at 4% APR, compounded continuously, to grow to \$2000 ?
We need to solve the equation for t.

$$2000 = 1000e^{.04t}$$

First, divide by 1000 on both sides to leave $e^{.04t}$ alone on the right side.

$$\begin{aligned} \frac{2000}{1000} &= \frac{1000e^{.04t}}{1000} \\ 2 &= e^{.04t} \end{aligned}$$

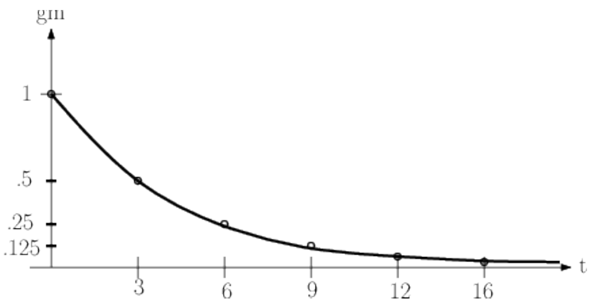
The trick is to log both sides of the equation, and here we use the natural log because of the convenient identity $\ln e^x = x$.

$$\begin{aligned} \ln 2 &= \ln e^{.04t} \\ \ln 2 &= .04t \\ \frac{\ln 2}{.04} &= t \\ t &\approx 17.33\, years \end{aligned}$$

Example What interest rate is needed for \$5000 to grow to \$20,000 in 22 years?

$$\begin{aligned} 20000 &= 5000e^{r22} \\ \frac{20000}{5000} &= e^{22r} \\ 4 &= e^{22r} \\ \ln 4 &= 22r \\ r &= \frac{\ln 4}{22} \approx .063 = 6.3\% \end{aligned}$$

Radioactive Decay



A radioactive isotope of an element decays in proportion to the amount. Its rate of decay is often expressed as its half life, the time required for half of the radioactive isotope to decay. Half lives of elements variy between 4.5 billion years for U_{238}(Uranium 238) to Mg_{34}(Magnesium 34) with a half life of 20 milliseconds.

The convenient form of the decay equation is

$$y = y_0e^{kt}$$

y = amount after t years
y_o= amount when t = 0
t = time elapsed, units vary from years to microseconds
k = the decay constant for the particular isotope

The decay constant k is calculated as

$$k = \frac{\ln .5}{half - life}$$

For U_{238}, $k = \frac{\ln .5}{4.5} = -.1540$
For Mg_{34}, $k = \frac{\ln .5}{20} = -.03466$
Note the difference in units, between billions of years for U_{238} and milliseconds for Mg_{34}.

Example How much of 1 kg of U_{238} will be left after 2 billion years?
We already found that k = -.1540, so

$$y = 1e^{-.1540(2)} = .735\, kg$$

Example Au_{195}(Gold 195) has a half life of 186.09 days. How long till a .74 oz gold coin of Au_{195}decays till there is only .05 oz of gold left?
First, find k: $k = \frac{\ln .5}{186.09} = -.003725$
Then, solve the decay equation for t:

$$\begin{aligned} y &= y_0e^{kt} \\ .05 &= .74e^{-.003725t} \end{aligned}$$

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\begin{aligned}
\frac{.05}{.74} &= e^{-.003725t} \\
.06757 &= e^{-.003725t} \\
\ln .06757 &= -.003725t \\
t &= \frac{\ln .06757}{-.003725} = 723.38 \; days
\end{aligned}
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Exercises

1. How long till \$1000 invested at 5% APR, compounded continuously, grows to \$2500 ?
2. At what interest rate will \$5000 grow to \$10,000 in 18 years?
3. How long till 3 mg of iodine 131 decays to 1 mg? Its half life is 8.07 days.
4. How long till a 350% level of Radon 222 decay below 100%. Here, 100% is the maximum safe level. Its half life is 3.82 days.