0.1 Rules of Logarithms

Logarithms have been the cornerstone of scientific, engineering, and technical calculations starting in the early 1600s when logarithm tables and calculating devices (slide rules and similar logarithmic measuring devices) spread throughout Europe to around 1975 when hand scientific calculators became cheap. The cheap scientific calculator wiped out the slide rule industry, and tables of logarithms rarely appear in books for anything other than historical interest.

However, logarithms remain important for myriad reasons besides aiding routine computation, so it may be helpful to explore those properties of logarithms which were so useful for over 300 years. In fact, some of these calculation methods remain important for engineers who design computing hardware.

Rules

Logarithms are effectively treated as exponents (to the base of the log), and thus rules of exponents apply.

$$\begin{aligned} \log_b AB &= \log_b A + \log_b B \\ \log_b \frac{A}{B} &= \log_b A - \log_b B \\ \log_b A^r &= r \cdot \log_b A \end{aligned}$$

Algebraic Manipulation using Logarithms

Example Expand $\log \frac{x^4 y^{12}}{\sqrt[4]{z^3}}$ to an expression containing only logx, logy, and logz.
$\log \frac{x^4 y^{12}}{\sqrt[4]{z^3}} = \log x^4 y^{12} - \log z^{3/4} = \log x^4 + \log y^{12} - \frac{3}{4}\log z = 4\log x + 12\log y - \frac{3}{4}\log z.$

Example Combine the following expression into a single logarithm. $\log x + 2\log(x-1) - 3\log(2x+1)$
$\log x + 2\log(x-1) - 3\log(2x+1) = \log x + \log(x-1)^2 - \log(2x+1)^3 = \log \frac{x(x-1)^2}{(2x+1)^3}.$

Examples Do the following.
Expand:$\log x^2 y^3 = \log x^2 + \log y^3 = 2\log x + 3\log y$
Expand: $\log \frac{x}{\sqrt{z}} = \log x = \log \sqrt{z} = \log x - \log z^{\frac{1}{2}} = \log x - \frac{1}{2}\log z$
Combine: $\log x + 4\log y - 3\log z = \log x + \log y^4 - \log z^3 = \log \frac{xy^4}{z^3}$

Since the log and exponential functions are inverses of eachother,

$$\log_b b^x = x \qquad b^{\log_b x} = x$$

$\log_b b^x = x$is true for all x, but x must be positive for $b^{\log_b x} = x$ to be true.

Examples Do the following.
$10^{\log x} = x$
$\log 10^{3.7y} = 3.7y$
$\ln e^{-1.2t} = -1.2t$
$e^{\ln 4.7} = 4.7$

Multiplication and Division using Logarithms

Logarithm tables listed the log of numbers in a certain fairly small range, and this range was from 1 to 10 for the common log. The common log uses base 10.

$$\log_{10} x \equiv \log x$$

Here is a crude 2-digit table of common logs.

	.0	.1	.2	.3
1	.0000	.0414	.0792	.1139
2	.3010	.3222	.3424	.3617
3	.4771	.4914	.5051	.5185
4	.6021	.6128	.6232	.6335
5	.6990	.7076	.7160	.7243
6	.7782	.7853	.7924	.7993
7	.8451	.8513	.8573	.8633
8	.9030	.9085	.9138	.91919
9	.9542	.9590	.9638	.9685

Example Calculate (2.3)(3.1) using the table of logs.

log(2.3) = .3617, and log(3.1) = .4914 .

Thus, log((2.3)(3.1)) = log(2.3) + log(3.1) = .3617 + .4914 = .8531 .

Then, the antilog of .8531 is found by scanning logs in the table and finding the closest, and hence, the closest decimal number which is the antilog. Thus, antilog(.8531) = 7.1 . Now, (2.3)(3.1) = 7.13 exactly, and this 2-digit multiplication calculation is probably easier than dealing with logarithms. However, table lookups as well as simple additions and subtractions become attractive when there are more significant digits with accompanying larger and more accurate tables of logarithms.

Roots and Powers using Logarithms

The core principle of using logarithms for calculation is to take the log of a number, use properties of logarithms for calculation, then take the antilog to find the result. This works quite well for exponentiation also.

Example Calculate 1.2^{30}using the table of logarithms.
$\log 1.2^{20} = 20 \cdot \log 1.2$. From the table, log(1.2) = .0792 . Then 20(.0792) = 1.584 . Then, antilog(1.584) = antilog(1) ·antilog(.584) = 10 ·(3.8) = 38. This compares well to 4 significant digits obtained from a calculator 38.3376 , and it remains sufficient for a rough approximation.

Roots and powers are natural for logarithms. There were also log-trig tables available to create shortcuts for those people doing trigonometric calculations, say navigators and surveyors. Those days are gone, yet some of these tricks are still useful.

Example What is 9.3^{2000}? This exceeds the range for most scientific calculators though it could done using a symbolic math program on a computer. However, $\log 9.3^{2000} = 2000 \cdot \log 9.3 = 2000 \cdot (.9685) = 1937$. Then, antilog(1937) = 10^{1937} for a nice estimate.

Example Calculate $\sqrt[3]{8.8}$. Now, $\sqrt[3]{8.8} = 8.8^{1/3}$. $\log 8.8^{1/3} = \frac{1}{3}\log 8.8 = \frac{1}{3}(.9445) = (.9445) \div 3 = .3148$. The, antilog(.3148)=2.1 . This compares to a 4 decimal place calculator result of 2.0646 .

Exercises

Expand the following using only $\log x$, $\log y$, and $\log z$.

1. $\log_2 x^4y^2$
2. $\log \sqrt{xy^3}$
3. $\log \frac{x}{x^5}$
4. $\ln \frac{x^2y}{z^4}$

Combine the following into a single log.

5. $\log x + \log y - \log z$
6. $4\log x - 7\log y$
7. $\frac{1}{3}\log x + \frac{2}{3}\log y$
8. $\log x - \frac{1}{2}\log x - \log y$