1 Exponents and Logarithms

1.1 Exponential Functions

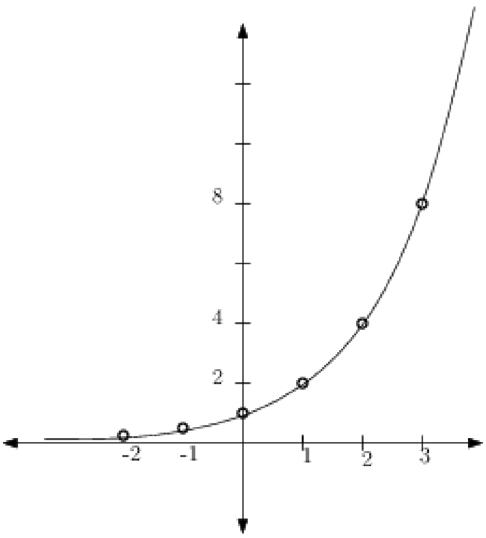
Polynomial functions such as $g(x) = x^2$ have fixed exponents. When the exponent itself includes a variable, such as $f(x) = 2^x$, we have an exponential function.

$$f(x) = a^x \qquad (a > 0, a \neq 1)$$

The base for this general exponential function is a, and the exponent is x.

The exponential function $f(t) = 2^t$has a base of 2 and an exponent of t. It is usual to use t as the variable when we have a function of time.

The graph of this function shows the doubling every hour.



Example A lactobacillus acidophilus bacteria culture in milk (making yogurt here) doubles in population every hour, and the culture is started with 1000 bacterium. How many bacterium are in the culture after 10 hours?

The amount of bacteria present is modeled by the function $f(t) = 2^t$where t is the time in hours after the culture is started, and f(t) is the number of bacterium in thousands. Thus, $f(10) = 2^{10} = 1024$, so there are 1024 thousand bacterium after ten hours.

Exponential functions occur when the rate of change of a value is directly related to the value.

Compound Interest

For example, an annual percentage interest rate of 2% (compounded annually) on a savings account certificate will generate interest depending on how much money is in the account. A certificate of \$1000 will generate \$20 of interest, and a certificate of \$10,000 will generate \$200 of interest. Thus, the more you have the more you get.

After one year, we have: $1000 + 20 = 1020$
For the second year, we now earn interest on 1020, so we get: $1020 + .02(1020) = 1040.40$
For the third year, we get: $1040.40 + .02(1040.40) = 1061.21$
There is a shortcut to making an annual calculation.

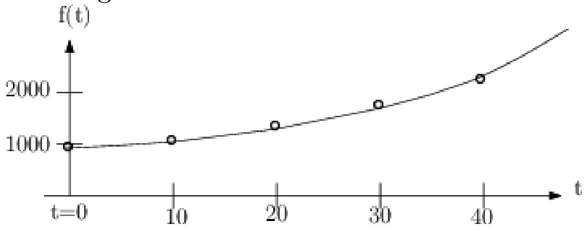
The amount of money in this saving certificate after t years can be modeled by the equation

$$f(t) = 1000 \cdot (1.02)^t$$

For t = 0, we have $f(0) = 1000 \cdot (1.02)^0 = 1000 \cdot 1 = 1000$.
For t = 1, we have $f(1) = 1000 \cdot (1.02)^1 = 1000(1.02) = 1020$. Here we have earned \$20 in interest.
Here is a table showing the value of the certificate for values of t.

t	f(t)
0	1000.00
1	1020.00
2	1040.40
3	1061.21
4	1082.43
10	1218.99
20	1485.95
30	1811.36
40	2208.04
50	2691.59

If we graph this function f(t), the increasing rate of growth is clearer.



What happens if interest in compounded four times a year, or daily?

Consider the situation above where 1000 is deponsited at 2% annual interest, but, instead of compounding the interest after each year, we coumpound it after each qrarter. the interest per quarter is $\frac{2\%}{4} = .5\% = .005$. Instead of calculating the interest each quarter, we can condense compound interest calculation with the formula

$$f(t) = P\left(1 + \frac{i}{k}\right)^{kt}$$

where P is the initial investment, i is the annual interest rate as a decimal, t is the time in years, and k is the number of compounding periods per year.

For an investment of 1000 at 2%=.02, compounded quarterly, for 3 years, we obtain

$$\begin{aligned} f(4) &= 1000\left(1 + \frac{.02}{4}\right)^{4\cdot 3} \\ &= 1000(1.005)^{12} = 1061.68 \end{aligned}$$

Here \$1061.68 is slightly larger than \$1061.21 which resulted from annual compounding of interest.

The Natural Number $e \approx 2.718281828$

The natural number is irrational, greater than 2 and less than 3, fairly close to 2.7 . It occurs naturally in a great many mathematical problems, hence it is called the natural number. You do not need to memorize it to great precision since scientific calculators already use it. We will be raising e to powers, and there should be a "e" button on your calculator above the "ln" button (where "ln" means the natural log, which we will look at later.)

Consider the problem above in computing the value of an investment with compound interest. When interest is compounded not quarterly, daily, or even hourly, but continuously, the compound interest formula simplifies to

$$f(t) = Pe^{rt}$$

For example, \$1000 invested at 2% compounded continuously for 3 years would become

$$f(3) = 1000e^{(.02)3} = 1000e^{.06} = 1061.84$$

\$1061.84 is slightly greater than the \$1061.68 earned from quarterly compounding of interest.

Example Radioactive elements decay exponentially. The isotope N_{16} (nitrogen 16) is continually produced within water-cooled nuclear reactors by bomardment of O_{16}(oxygen 16) by neutrons, and N_{16} has a half-life of 7.35 seconds. This means that a half a quantity of N_{16} will decay in 7.35 seconds.

The decay equation for N_{16} is

$$f(t) = q_0 e^{-.0943t}$$

where q_0 is the amount of N_{16} at time t = 0, t is the time elapsed in seconds, and f(t) is the amount left after t seconds.

How much of 4 micrograms of N_{16} will be left after 1 minute?
Here, t = 60 (1 minute = 60 seconds), so

$$f(60) = 4e^{-.0943(60)} = 4e^{-5.658} = 4(.00349) = .014$$

Thus, there would be .014 micrograms of N_{16} left after 1 minute.

Exercises

The natural number e, to 20 places, is 2.71828 18284 59045 23536 . The natural number e pops up in many situations which, at first glance, might appear unrelated. This is the charm of mathematics and the natural sciences. e can be generated in many ways, and here are a few ways below.

1. $e = \lim_{x\to\infty}\left(1 + \frac{1}{x}\right)^x$

Complete the table:

x	$\left(1+\frac{1}{x}\right)^x$
10	$(1+1/10)^{10} = 2.5937$
100	
1000	
10000	
100000	
1000000	

2. $e = 2 + \frac{1}{1\cdot 2} + \frac{1}{1\cdot 2\cdot 3} + \frac{1}{1\cdot 2\cdot 3\cdot 4} + \cdots$

Complete the table:

number of terms	estimate
1	2
2	$2 + \frac{1}{1\cdot 2} = 2.5$
3	
4	
5	
6	

Evaluate the following.

3. $10^{2.7}$
4. $5.12^{6.02}$
5. $0.0032^{-12.72}$
6. $312.9^{-2.05}$
7. Suppose that an ecoli bacteria culture doubles in population every 17 minutes. Then, for t = the numbers of minutes elapsed, $\frac{t}{17}$ would be the number of times the ecoli culture has doubled in population.
   We then have the ecoli population increased by a factor of $2e^{\frac{t}{17}}$ each t minutes. If the population at time t = 0 is $q_o = 1000$, the population at time t is then

   $$q(t) = q_o 2^{\frac{t}{17}}$$

   What is the population at time t = 200 ?

Suppose that $10,000 is invested at 6% annual interest for 10 years. How much will the investment grow to if it is compounded

8. annually?
9. daily?
10. continuously?
11. Radon (Rn_{222}) has a half-life of 3.8235 days, and its decay equation is

    $$f(t) = q_0 e^{-.1813t}$$

    If 10 millicuries of Radon are present, how much is left after four weeks?