0.1 Algebra of Functions

Functional notation is designed to show action. The definition of a function, say,

$$f(x) = x^2 + 4x + 6$$

allows us to "f" things like numbers. "f of 5" means

$$f(5) = 5^2 + 4 \cdot 5 + 6 = 25 + 20 + 6 = 51$$

and so we write $f(5) = 51$.

You will always have a function definition to use when you need to apply a function. There is nothing special or sacred about the variable (usually x) used in a function definition. the function definition simply defines what the function does. The function $f(x) = x^2+4x+6$ says that when you "f" a number, take the sum of the number squared, the number times four, and six.

Letters f, g, and h are commonly used for function, but any letter will do so long as the meaning is clear.

Example $g(x) = \sqrt{3x+7}$
$g(2) = \sqrt{3 \cdot 2 + 7} = \sqrt{6+7} = \sqrt{13}$
$g(-1) = \sqrt{3 \cdot (-1) + 7} = \sqrt{-3+7} = \sqrt{4} = 2$

Example $h(x) = \dfrac{2x}{x^2 - 1}$
$h(4) = \dfrac{2 \cdot 4}{4^2 - 1} = \dfrac{8}{15}$

Function Arithmetic

You can add, subtract, multiply, and divide functions to form new functions.

$(f+g)(x) = f(x) + g(x)$

$(f-g)(x) = f(x) - g(x)$

$(fg)(x) = f(x) \cdot g(x)$

$\left(\dfrac{f}{g}\right)(x) = \dfrac{f(x)}{g(x)}$

Example Consider the functions: $f(x) = x^2$ $\quad g(x) = 5x - 1 \quad h(x) = \dfrac{2}{x}$
$(f+g)(7) = f(7)+g(7) = 7^2+(5\cdot 7-1) = 49 + 34 = 83$
$\left(\dfrac{h}{g}\right)(2) = \dfrac{h(2)}{g(2)} = \dfrac{2/2}{5 \cdot 2 - 1} = \dfrac{1}{19}$

Composition

Composition of functions is taking the function of a function's result. The traditional notation is the circle meaning *compose*: $\circ$
$f \circ g$ reads as "f compose g". In practice, one usually applies a composition of functions to a number or expression, and it is also usual to eliminate the compose symbol in this fashion.

$$(f \circ g)(x) = f\,(g(x))$$

Here, $f\,(g(x))$ reads "f of g of x. This is a situation where action is taken right to left. For $f\,(g(x))$, one finds $f(x)$ first, then $g(f(x))$.

Example Consider the functions $f(x) = x^2$ and $g(x) = 3x + 2$
$(f \circ g)(4) = f(g(4)) = f(3 \cdot 4 + 2) = f(14) = 14^2 = 196.$
$(g \circ f)(4) = g(f(4)) = g(4^2) = g(16) = 3 \cdot 16 + 2 = 50.$

$$\begin{aligned}
(f \circ g \circ g)(-1) &= f(g(g(-1))) \\
&= f(g(\ 3(-2) + 2)\ ) \\
&= f(g(-4)) \\
&= f(\ 3(-4) + 2\ ) \\
&= f(-10) \\
&= (-10)^2 = 100
\end{aligned}$$

Exercises

Consider the functions:
$f(x) = 2x - 5$
$h(x) = x^2 + x$
$g(x) = \dfrac{2}{x}$

1. (f + g)(5) =
2. f(g(1)) =
3. g(f(1)) =
4. f(h(3)) =
5. g(h(-4)) =
6. f(f(6)) =
7. f(g(-4)) =
8. $(h \circ f)(4) =?$
9. $(f \circ f \circ f)(5) =?$