0.1 Completing the Square

Recall the perfect square formulas

$$\begin{aligned}(p+q)^2 &= p^2+2pq+q^2\\(p-q)^2 &= p^2-2pq+q^2\end{aligned}$$

We want to turn trinomials of the form Ax^2+Bx+C into perfect squares, but, to simplify matters, we always want to factor out the coefficient of the Ax^2 term and work on the simpler trinomial

$$x^2+bx+c$$

This is not usually a perfect square, but, it turns out that we can always add or subtract a number to x^2+bx+c so that we do have a perfect square. Consider the equation

$$x^2+bx+c = 0$$

It is usually easiest to add the opposite of c to both sides, then figure out the number needed to make the perfect square on the left.

$$\begin{aligned}x^2+bx+c &= 0\\ -c \quad & \quad -c\\ \hline x^2+bx &= -c\end{aligned}$$

At this point, the number needed is $\left(\frac{b}{2}\right)^2$ which we add to both sides.

$$\begin{aligned}x^2+bx+\frac{b^2}{4} &= -c+\frac{b^2}{4}\\(x+\frac{b}{2})^2 &= -c+\frac{b^2}{4}\end{aligned}$$

This process works easiest when b is an even number, and this is the case we will focus on.

Example Complete the square to solve $x^2+6x+5=0$

$$\begin{aligned}x^2+6x+5 &= 0\\ -5 \quad & \quad -5\\ \hline x^2+6x &= -5\end{aligned}$$

Here b = 6, so $\left(\frac{b}{2}\right)^2 = \left(\frac{6}{2}\right)^2 = 3^2 = 9$, and then we add 9 to both sides to obtain a perfect square on the left side.

$$\begin{aligned}x^2+6x &= -5\\ 9 \quad & \quad 9\\ \hline x^2+6x+9 &= 4\\(x+3)^2 &= 4\end{aligned}$$

At this point, we can take the square root of both sides, then solve for x. Note the $\pm$ put in front of the radical on the right side.

$$\begin{aligned}(x+3)^2 &= 4\\x+3 &= \pm\sqrt{4}\\x+3 &= \pm 2\\ -3 \quad & \quad -3\\ \hline x &= -3\pm 2\end{aligned}$$

Thus, our solutions are -3 ± 2 which means the solution set $\{-3+2,-3-2\}=\{-1,-5\}$.

Example Solve $x^2-8x+1=0$.

$$\begin{aligned}x^2+8x+1 &= 0\\ -1 \quad & \quad -1\\ \hline x^2+8x &= -1\\x^2+8x+\left(\frac{8}{2}\right)^2 &= -1+\left(\frac{8}{2}\right)^2\\x^2+8x+16 &= 15\\(x+4)^2 &= 15\\x+4 &= \pm\sqrt{15}\\ -4 \quad & \quad -4\\ \hline x &= -4\pm\sqrt{15}\end{aligned}$$

Exercises

Solve each of the following by completing the square.

1. $x^2-6x-1=0$
2. $x^2+24x+101=0$
3. $x^2+3x+1=0$
4. $x^2-4x+1=0$
5. $x^2+6x+20=0$
6. $x^2-8x-2=0$