0.1 Complex Numbers

The solution to the equation

$$x^2 = -1$$

cannot be 1 or -1 because

$$\begin{aligned} 1^2 &= 1 \\ (-1)^2 &= 1 \end{aligned}$$

For hundreds of years mathematicians considered such equations impossible and left the matter at that. Some proposed a special number that when squared would be negative. It could not be a real number because any real number squared is positive or zero in the case of $0^2 = 0$. These quantities were labeled imaginary because they were not real, and the name stuck. We only need one such special number, however, to build all the rest as needed.

We define i to be $\sqrt{-1}$ with the property that $i^2 = -1$, or $\left(\sqrt{-1}\right)^2 = -1$.

$$\begin{aligned} i &= \sqrt{-1} \\ i^2 &= -1 \end{aligned}$$

Thus, we do have solutions to the equation $x^2 = -1$, namely i and -i because

$$\begin{aligned} i^2 &= -1 \\ (-i)^2 &= (-i)(-i) = i^2 = -1 \end{aligned}$$

Our properties of radicals allow us to handle any square root of a negative number by factoring out the -1.

Example Find the following square roots.

1. $\sqrt{-4} = \sqrt{4(-1)} = \sqrt{4}\sqrt{-1} = 2i$
2. $\sqrt{-7} = \sqrt{7(-1)} = \sqrt{7}\sqrt{-1} = \sqrt{7}i$
3. $\sqrt{-18} = \sqrt{18}\sqrt{-1} = \sqrt{9 \cdot 2}i = \sqrt{9}\sqrt{2}i = 3\sqrt{2}i$.

Sometimes it is easy for the i to get lost under the radical, so it is common to put the i in front of the radical, so the latter two examples we might write as

$$\begin{aligned} \sqrt{7}i &= i\sqrt{7} \\ 3\sqrt{2}i &= 3i\sqrt{2} \end{aligned}$$

In general for square roots of negative numbers

$$\begin{aligned} a &> 0 \Rightarrow \\ \sqrt{-a} &= i\sqrt{a} \end{aligned}$$

Complex Numbers

A complex number has a real part and an imaginary part, and we use the form

$$a + bi \qquad a, b \in \mathbb{R}$$

The following are complex numbers:

Complex Number	Real Part	Complex Part
2+3i	2	3
6-9i	6	-9
$1 - i\sqrt{5}$	1	$-\sqrt{5}$
18i	0	18
9	9	0

Arithmetic

Complex numbers can be added, subtracted, multiplied, and divided. Addition, subtraction, and multiplication are most straightforward and similar to working with polynomials. Consider complex numbers $a + bi$ and $c + di$.

$$\begin{aligned} (a+bi)+(c+di) &= (a+c)+(b+d)i \\ (a+bi)-(c+di) &= (a-c)+(b-d)i \\ (a+bi)(c+di) &= (ac-bd)+(ad+bc)i \end{aligned}$$

Examples Perform the arithmetic operations and simplify to the form a + bi.

1. (2+8i) + (6-2i) = (2+6) + (8-2)i = 8 + 6i
2. $(3-5i)-(7+i) = (3-7)+(-5-1)i = -4-6i$
3. $6(5-2i) = 6 \cdot 5 - 6(2i) = 30 - 12i$
4. $3i(7-5i) = 3i \cdot 7 - 3i \cdot 5i = 21i - 15i^2 = 21i - 15(-1) = 15 + 21i$

5. $(6+5i)(2-7i) = 6\cdot 2-6\cdot 7i+5i\cdot 2-5i\cdot 7i = 12 - 42i + 10i - 35i^2 = 12 + 35 - 32i = 47 - 32i$

Note that in the last example, the two binomials were simply FOILed out which is just the distributive law.

Complex Conjugates

Recall that $(a+b)(a-b) = a^2 - b^2$, and recall the trick for rationalizing the denominator of a fraction like $\frac{1}{\sqrt{3}+\sqrt{2}}$.

$$\begin{aligned}(\sqrt{3}+\sqrt{2})(\sqrt{3}-\sqrt{2}) &= \left(\sqrt{3}\right)^2 - \left(\sqrt{2}\right)^2 = \\ &= 3 - 2 = 1\end{aligned}$$

Similarly for complex numbers

$$\begin{aligned}(a+bi)(a-bi) &= a^2 + abi - abi - b^2i^2 \\ &= a^2 + b^2\end{aligned}$$

In fact, for exvery complex number $a+bi$, there is a conjugate $a - bi$ so that $(a+bi)(a-bi) = a^2 + b^2$. $a^2 + b^2$is a positive real number (zero if a and b are both zero).

Examples Find the conjugate of the following numbers.

1. 2-7i
   The conjugate is 2 + 7i.
   Likewise, the conjugate of 2 + 7i is 2 - 7i.
2. 8 + 11i
   The conjugate is 8 - 11i.
   Likewise, the conjugate of 8 - 11i is 8 + 11i.
3. 5i
   The conjugate is -5i, and the conjugate of -5i is 5i
4. 2
   The conjugate is 2. the conjugate of any real number is itself, a real number. 2 is a complex number, and you can think of it as 2 = 2 + 0i, then the conjugate of 2 = 2 + 0i is 2 - 0i = 2.

Division

Division by a complex number is different because ordinary long division will not work. We can divide a complex numebr by a non-zero real number in the ordinary way, however.

$$(7+8i) \div 3 = \frac{7+8i}{3} = \frac{7}{3} + \frac{8}{3}i$$

Consider the problem $3 \div (3+5i)$. the way we do this multiply the numerator and denominator by the complex conjugate of the denominator. Then the denominator will be a positive real number, and we can split out the real and imaginary parts for the answer in the form a + bi.

Example Divide: $3 \div (3+5i)$

$$\begin{aligned}3 \div (3+5i) &= \frac{3}{3+5i} \\ &= \frac{3}{3+5i} \cdot \frac{3-5i}{3-5i} \\ &= \frac{3(3+5i)}{(3+5i)(3-5i)} \\ &= \frac{9+15i}{3^2+5^2} \\ &= \frac{9+15i}{34} \\ &= \frac{9}{34} + \frac{15}{34}i\end{aligned}$$

Exercises

Perform any operations then simplify.

1. $\sqrt{-12}$
2. $\sqrt{-18} + \sqrt{36}$
3. $(3-5i) - (6-7i)$
4. $(2i)(-6i)$
5. $3i(4+10i)$
6. $(2+9i)(5-3i)$
7. $\frac{1}{2+6i}$
8. $\frac{1+2i}{3-7i}$
9. $3i(6-7i)$
10. $(2+3i)(2-3i)$
11. $\frac{3}{4-5i}$
12. $\frac{3+i}{4+7i}$
13. $\sqrt{-50}$