0.1 Radical Operations

Simplifying

One could use exponent forms rather than radicals in all cases, yet radicals are still commonly used. To be practical, the square root is by far the most common radical form, cube roots much less common, and higher index roots like $\sqrt[7]{x}$ quite rarely seen. Nevertheless, rules for radicals apply to all radicals–not just the square roots and maybe a few cube roots that you are likely to see in your life–so we might as well make the strong radical statements.

$$\begin{aligned} \sqrt[n]{x}\,\sqrt[n]{y} &= \sqrt[n]{xy} \\ \frac{\sqrt[n]{x}}{\sqrt[n]{y}} &= \sqrt[n]{\frac{x}{y}} \end{aligned}$$

In words, the first rule says that "The product of roots is the root of the product". The second rule says that "The quotient of roots is the root of the quotient."

Examples Simplify the following

1. $\sqrt{8}\sqrt{2} = \sqrt{8 \cdot 2} = \sqrt{16} = 4$
2. $\dfrac{\sqrt{50}}{\sqrt{2}} = \sqrt{\dfrac{50}{2}} = \sqrt{25} = 5$

To be a bit more useful, we point out here that the nth root of an nth power of an expression is the expression.

$$\sqrt[n]{a^n} = a$$

Also,

$$\left(\sqrt[n]{a}\right)^n = a$$

These are just the rules of exponents written in radical form.

$$\begin{aligned} \sqrt[n]{a^n} &= (a^n)^{\frac{1}{n}} = a^{n \cdot \frac{1}{n}} = a^{\frac{n}{n}} = a \\ a &= a^{\frac{n}{n}} = a^{\frac{1}{n} \cdot n} = \left(a^{\frac{1}{n}}\right)^n = \left(\sqrt[n]{a}\right)^n \end{aligned}$$

There is one reservation about these rules above, and this is that $\sqrt[n]{x^n} = x$ is false when n is even and x is less than zero. For some examples, $\sqrt{(-2)^2} = \sqrt{4} = 2$, and $\sqrt[4]{(-3)^4} = \sqrt[4]{81} = 3$. In the case when x is negative and n is even, $\sqrt[n]{x^n} = |x|$. In fact, $\sqrt[n]{x^n} = |x|$ is always true, and $\sqrt{x^2}=|x|$ is one practical way of obtaining the absolute value of a number. We do assume, however, that variables are positive in our exercises and examples where it could matter while we are working on simplifying radicals. When it comes to radical equations, we will revisit this issue more completely.

Examples Simplify the following.

1. $\sqrt[3]{x}\sqrt[3]{x^2} = \sqrt[3]{x^2 x} = \sqrt[3]{x^3} = x$
2. $\dfrac{\sqrt[5]{x^8}}{\sqrt[5]{x^3}} = \sqrt[5]{\dfrac{x^8}{x^3}} = \sqrt[5]{x^5} = x$

We can also use rules of exponents to simplify roots for which the power does not equal the index.

Examples Simplify the following.

1. $\sqrt[3]{x^5}\sqrt[3]{x^7} = \sqrt[3]{x^5 x^7} = \sqrt[3]{x^{12}} = x^{\frac{12}{3}} = x^4$
2. $\dfrac{\sqrt[4]{x^{17}}}{\sqrt[4]{x^5}} = \sqrt[4]{\dfrac{x^{17}}{x^5}} = \sqrt[4]{x^{12}} = x^{\frac{12}{4}} = x^3$

In general, for $\sqrt[n]{x^m}$, if n divides m evenly, $\dfrac{m}{n}$ will be an integer, so we can simply write the simplified result as

$$\sqrt[n]{x^m} = x^{\frac{m}{n}}$$

Examples Simplify $\sqrt[4]{x^{24}}$.

$$\sqrt[4]{x^{24}} = x^{\frac{24}{4}} = x^6$$

What if for $\sqrt[n]{x^m}$ that n does not divide m? To leave the result in radical form, we take the root of that part of m which is divisible by n, and leave the remainder as the exponent of x in the radical.

Example Simplify the following.

1. $\sqrt[3]{x^5}=\sqrt[3]{x^3 x^2} = \sqrt[3]{x^3}\sqrt[3]{x^2} = x\sqrt[3]{x^2}$
   Because 3 goes into 5 one time with a remainder of 2.
2. $\sqrt[5]{y^{23}} = \sqrt[5]{y^{20}y^3} = \sqrt[5]{y^{20}}\sqrt[5]{y^3} = y^4\sqrt[5]{y^3}$
   because 5 goes into 23 4 times with a remainder of 3.

For simplifying more complicated radicals, we can use the radical multiplication and division rules to take the root of as much as possible.

Examples Simplify the following.

1. $\sqrt[3]{x^6 y^{12}} = \sqrt[3]{x^6}\sqrt[3]{y^{12}} = x^2 y^4$
2. $\sqrt[5]{x^{35}y^{17}} = \sqrt[5]{x^{35}}\sqrt[5]{y^{17}} = x^7\sqrt[5]{y^{15}y^2} = x^7\sqrt[5]{y^{15}}\sqrt[5]{y^2} = x^7 y^3\sqrt[5]{y^2}$
3. $\sqrt{8x^5y^8} = \sqrt{4x^4y^6}\sqrt{2x} = 2x^2y^3\sqrt{2x}$

Arithmetic

Like radicals can be added and subtracted as if they were variables themselves, and “like” here means exactly the same, index and radicand both.

$$\begin{aligned}\sqrt{x}+\sqrt{x} &= 2\sqrt{x}\\ 3\sqrt[4]{y^3}+7\sqrt[4]{y^3} &= 10\sqrt[4]{y^3}\\ 3\sqrt[3]{5xy^2}-8\sqrt[3]{5xy^2} &= -5\sqrt[3]{5xy^2}\end{aligned}$$

Radicals can be multiplied and divided, as shown earlier, and so complicated expressions of radicals can be simplified.

Examples Simplify the following.

1. $3(\sqrt{5}-8)=3\sqrt{5}-24$
2. $\sqrt{2}(\sqrt{6}-\sqrt{2})=\sqrt{2}\sqrt{6}-\sqrt{2}\sqrt{2}=\sqrt{12}-2=\sqrt{4\cdot 3}-2=2\sqrt{3}-2$
3. $(\sqrt{2}-\sqrt{5})(\sqrt{6}+\sqrt{5})=\sqrt{2}\sqrt{6}+\sqrt{2}\sqrt{5}-\sqrt{5}\sqrt{6}-\sqrt{5}\sqrt{5}=\sqrt{12}+\sqrt{10}-\sqrt{30}-5=2\sqrt{3}+\sqrt{10}-\sqrt{30}-5$

Rationalizing the Denominator

There is nothing really wrong with having a radical in the denominator, like $\frac{1}{\sqrt{2}}$. In earlier days people would find an approximation to $\sqrt{2}$ like 1.414, but the long division problem $\frac{1}{1.414}$ is tedious. Instead,

$$\frac{1}{\sqrt{2}}=\frac{1}{\sqrt{2}}\cdot\frac{\sqrt{2}}{\sqrt{2}}=\frac{\sqrt{2}}{2}$$

then $\frac{1.414}{2}$ is a relatively easy long division problem, and the result is .707 . Many algebra and math instructors insist on rationalizing the denominator for square roots, yet this insistence is not here. Rather, it is important to know how to rationalize the denominator, the technique, because it is also important sometimes to rationalize the numerator. In general, you multiply the numerator and denominator by the square root which occurs in the denominator in order to rationalize the denominator:

$$\frac{a}{\sqrt{b}}=\frac{a}{\sqrt{b}}\cdot\frac{\sqrt{b}}{\sqrt{b}}=\frac{a\sqrt{b}}{b}$$

Another denominator to rationalize is in this fraction $\frac{3}{\sqrt{2}+5}$. We note from the formular for the difference of two squares that

$$(a+b)(a-b)=a^2-b^2$$

Similarly

$$(\sqrt{2}+5)(\sqrt{2}-5)=\left(\sqrt{2}\right)^2-5^2=2-25=-23$$

and so

$$\begin{aligned}\frac{3}{\sqrt{2}-5} &= \frac{3}{\sqrt{2}-5}\cdot\frac{\sqrt{2}+5}{\sqrt{2}+5}\\ &= \frac{3(\sqrt{2}+5)}{(\sqrt{2}-5)(\sqrt{2}+5)}\\ &= \frac{3(\sqrt{2}+5)}{2-25}\\ &= -\frac{3(\sqrt{2}+5)}{23}\end{aligned}$$

Example Rationalize the denominator for the fraction $\frac{3+\sqrt{7}}{\sqrt{5}-\sqrt{2}}$
We will multiply numerator and denominator by $\sqrt{5}+\sqrt{2}$.

$$\begin{aligned}\frac{3+\sqrt{7}}{\sqrt{5}-\sqrt{2}} &= \frac{3+\sqrt{7}}{\sqrt{5}-\sqrt{2}}\cdot\frac{\sqrt{5}+\sqrt{2}}{\sqrt{5}+\sqrt{2}}\\ &= \frac{(3+\sqrt{7})(\sqrt{5}+\sqrt{2})}{(\sqrt{5}-\sqrt{2})(\sqrt{5}+\sqrt{2})}\\ &= \frac{3\sqrt{5}+3\sqrt{2}+\sqrt{7}\sqrt{5}+\sqrt{7}\sqrt{2}}{\left(\sqrt{5}\right)^2-\left(\sqrt{2}\right)^2}\\ &= \frac{3\sqrt{5}+3\sqrt{2}+\sqrt{35}+\sqrt{14}}{3}\end{aligned}$$

The results are not always “simplified” if one is thinking of somehow making the fraction look less complicated. This example becomes more complicated looking with four different radicals in the numerator. Yet, the denominator has been rationalized.

Exercises

Simplify.

1. $\sqrt[3]{x^{12}}$
2. $\sqrt[5]{x^{25}y^{60}}$
3. $\sqrt[3]{y^{14}}$
4. $\sqrt[6]{x^2y}+11\sqrt[6]{x^2y}$
5. $2\sqrt[3]{3a^4}-3a\sqrt[3]{81a}$

6. $\sqrt{a^3b}\sqrt{ab^4}$

7. $\sqrt[4]{x}\,\sqrt[5]{x}$

8. $\sqrt{12}+\sqrt{3}$

9. $\dfrac{4}{\sqrt{7}}$

10. $\dfrac{3}{\sqrt{5}-4}$

11. $\sqrt[3]{x^3y^6}$

12. $\sqrt[5]{32x^5y^7}$

13. $\sqrt{12}(\sqrt{3}-5)$

14. $\left(\dfrac{3x^4y^6}{27xy^{-9}}\right)^{\frac{-1}{3}}$

15. $\dfrac{1}{\sqrt{7}}$

16. $\dfrac{3}{\sqrt{3}-4}$