1 Radicals and exponents

1.1 Fractional Exponents and Radicals

We define the n-th root of a number a to be a number b such that $b^n = a$. The notation for the n-th root is a radical symbol with the n tucked into the Northwest corner: $\sqrt[n]{a}$. The radical symbol is the box itself. The value inside the box is the radicand. The number n for the n-th root is called the index of the radical.

$$\sqrt[n]{a} = b \iff b^n = a$$

The n-th root of something in this radical form is also defined to be the principal n-th root.

Square roots are special in that they are commonplace compared to 3d root, 4th roots, and any other n-th root with an index besides 2. We do not usually use an index of 2 for a square root because it is convention that a radical without an index is the square root or 2nd root. It is not incorrect, however, to put in the index of 2, and it is important to know that the index is 2 with a square root.

$$\sqrt{a} = \sqrt[2]{a}$$

N-th roots can also be written using exponents.

$$\sqrt[n]{a} = a^{\frac{1}{n}}$$

Nth Root of nth Powers

Examples Write the following radicals in exponent form.

1. $\sqrt[3]{x} = x^{\frac{1}{3}}$
2. $\sqrt{x} = x^{\frac{1}{2}}$
3. $\sqrt[5]{32x^{10}} = (32x^{10})^{\frac{1}{5}}$

Note in the example above, $\sqrt[5]{32x^{10}} = (32x^{10})^{\frac{1}{5}}$, that there is an implied parenthesis around the entire radicand.

Consider the problem of converting $\sqrt[3]{x^2}$ to exponent form.

$$\sqrt[3]{x^2} = \left(x^2\right)^{\frac{1}{3}}$$

According to our rules of exponents when a power is raised to a power, we multiply exponents, so

$$\begin{aligned} \sqrt[3]{x^2} &= \left(x^2\right)^{\frac{1}{3}} \\ &= x^{\frac{2}{3}} \end{aligned}$$

This suggests a general rule.

$$\sqrt[n]{a^m} = a^{\frac{m}{n}}$$

Note that the exponent m from a^m goes in the numerator, and the index n always goes in the denominator. Further, since $m \cdot \frac{1}{n} = \frac{m}{n} = \frac{1}{n} \cdot m$, we have $a^{\frac{m}{n}} = \left(a^{\frac{1}{n}}\right)^m = \left(\sqrt[n]{a}\right)^m$. In summary,

$$\sqrt[n]{a^m} = a^{\frac{m}{n}} = \left(\sqrt[n]{a}\right)^m$$

Both these forms, $\sqrt[n]{a^m}$ and $\left(\sqrt[n]{a}\right)^m$ are equivalent, but sometimes one form is more convenient than the other.

Examples Write the following radicals in exponent form.

1. $\sqrt[5]{x^7} = x^{\frac{7}{5}}$
2. $\sqrt{x^3} = x^{\frac{3}{2}}$

Example Write the following in radical form.

1. $x^{\frac{5}{3}} = \sqrt[3]{x^5} = \left(\sqrt[3]{x}\right)^5$ Either radical form is correct.
2. $y^{\frac{5}{2}} = \sqrt[2]{y^5} = \sqrt{y^5} = (\sqrt{y})^5$ Any of these three radical forms are correct, yet the latter two forms are more usual.

Simplifying Radicals and Fractional Exponents

Most radical simplication is easiest done in exponent form which then boils down to simplifying of fractions.

$\frac{6}{3} = 2$, $\frac{12}{8} = \frac{3}{2}$, and $\frac{5}{25} = \frac{1}{5}$, so $x^{\frac{6}{3}} = x^2$, $x^{\frac{12}{8}} = x^{\frac{3}{2}}$, and $x^{\frac{25}{5}} = x^{\frac{1}{5}}$. Thus, $\sqrt[3]{x^6} = x^{\frac{6}{3}} = x^2$, $\sqrt[8]{x^{12}} = x^{\frac{12}{8}} = x^{\frac{3}{2}} = \sqrt[2]{x^3} = \sqrt{x^3}$, and $\sqrt[5]{x^{25}} = x^{\frac{25}{5}} = x^{\frac{1}{5}} = \sqrt[5]{x}$.

Note that the radical is gone when the denominator evenly divides the numerator in fractional exponent form, as in $\sqrt[3]{x^6} = x^{\frac{6}{3}} = x^2$.

Examples Simplify the following radicals.

1. $\sqrt[4]{x^{20}} = x^{\frac{20}{4}} = x^5$
2. $\sqrt[3]{x^6y^{21}} = \left(x^6y^{21}\right)^{\frac{1}{3}} = x^{\frac{6}{3}}y^{\frac{21}{3}} = x^2y^7$
3. $\sqrt[4]{81} = \sqrt[4]{3^4} = 3^{\frac{4}{4}} = 3^1 = 3$
4. $\sqrt[3]{8x^{36}} = \left(8x^{36}\right)^{\frac{1}{3}} = \left(2^3x^{36}\right)^{\frac{1}{3}} = 2^{\frac{3}{3}}x^{\frac{36}{3}} = 2x^{12}$

Exercises

Write as a radical:

1. $y^{\frac{2}{5}}$
2. $x^{\frac{1}{7}}$
3. $(x^3y^5)^{\frac{1}{4}}$
4. $x^{\frac{-2}{3}}$

Write using exponents.

5. $\sqrt[5]{x}$
6. $\sqrt[3]{x^8}$
7. $\sqrt{x^5y^7}$
8. $-\sqrt[6]{x^2y^{12}}$

Simplify the following:

9. $x^{\frac{3}{5}}x^{\frac{-2}{5}}$
10. $(x^8y^{10})^{\frac{1}{2}}$
11. $\dfrac{x^3x^{\frac{2}{5}}}{x^{\frac{4}{3}}}$
12. $\sqrt[5]{x^{15}y^{40}x^{100}}$
13. $\sqrt[3]{8x^6y^{42}}$
14. Write $\sqrt[5]{x^4}$ as an exponential expression.
15. Write $5x^{\frac{7}{8}}$ as a radical expression.