1 Rational Expressions

1.1 Multiplication and Division of Rational Expressions

A rational expression is also called a fraction of polynomials. $\frac{3x+8}{x-1}$, $\frac{x^2-2x}{x^2-4x+4}$, and $\frac{3}{x^2-x-1}$ are rational expressions, each having a polynomial their numerator and denominator. Rational expressions are simplified, multiplied, divided, added, and subtracted almost exactly as done with fractions of integers.

Simplifying Rational Expressions

To simplify the fraction $\frac{18}{12}$, we factor the numerator and denominator completely, then cancel common factors between numerator and denominator.

$$\begin{aligned}\frac{18}{12} &= \frac{2\cdot 3\cdot 3}{2\cdot 2\cdot 3}\\ &= \frac{\cancel{2}\cdot\cancel{3}\cdot 3}{\cancel{2}\cdot 2\cdot\cancel{3}}\\ &= \frac{3}{2}\end{aligned}$$

To simplify the fraction $\frac{4x^3y}{6xy^2}$, we do the same thing.

$$\begin{aligned}\frac{4x^3y}{6xy^2} &= \frac{2\cdot 2\cdot x^3y}{2\cdot 3\cdot xy^2}\\ &= \frac{\cancel{2}\cdot 2\cdot \cancel{x}\cancel{y}^{2}}{\cancel{2}\cdot 3\cdot \cancel{x}\cancel{y^2}^{1}}\\ &= \frac{2xx}{3y}\\ &= \frac{2x^2}{3y}\end{aligned}$$

To simplify the fraction $\frac{x^2-2x}{x^2-4}$, we do the same thing.

$$\begin{aligned}\frac{x^2-2x}{x^2-4} &= \frac{x(x-2)}{(x-2)(x+2)}\\ &= \frac{x\cancel{(x-2)}}{\cancel{(x-2)}(x+2)}\\ &= \frac{x}{x+2}\end{aligned}$$

Example Simplify $\frac{x^2-36}{x-6}$.

$$\begin{aligned}\frac{x^2-36}{x-6} &= \frac{(x-6)(x+6)}{(x-6)}\\ &= \frac{\cancel{(x-6)}(x+6)}{\cancel{(x-6)}}\\ &= \frac{x+6}{1}\\ &= x+6\end{aligned}$$

Note two things about this example.

1. We put parenthesis around the denominator: $(x-6)$. This is not strictly necessary, yet it is a useful step to highlight that we are really cancelling out the quantity $(x-6)$ from both numerator and denominator.
2. When we cancelled out $(x-6)$ from numerator and denominator, we did not end up with zero or nothing in the denominator. We ended up with a 1 in the denominator.

Multiplying

Recall how to multiply two fractions of integers.

$$\frac{a}{b}\cdot\frac{c}{d}=\frac{ac}{bd}$$

The same method holds for fractions of any kind.

Example Multiply then simplify
$\frac{3x}{4y^2}\cdot\frac{-16xy}{9x^4}$

$$
\begin{aligned}
&= \frac{3x(-16xy)}{4y^2(9x^4)} \\
&= \frac{-3\cdot 2\cdot 2\cdot 2\cdot 2x^2y}{2\cdot 2\cdot 3\cdot 3x^4y^2} \\
&= \frac{-\cancel{3}\cdot\cancel{2}\cdot\cancel{2}\cdot 2\cdot 2\cancel{x^2}\cancel{y}}{\cancel{2}\cdot\cancel{2}\cdot\cancel{3}\cdot 3\cancel{x^4}\cancel{y^2}} \\
&= \frac{-2\cdot 2}{3x^2y} \\
&= -\frac{4}{3x^2y}
\end{aligned}
$$

Example Multiply then simplify $\dfrac{x+4}{x^2-25}\cdot\dfrac{x^2-5x}{x^2+8x+16}$

$$
\begin{aligned}
&= \frac{x+4}{x^2-25}\cdot\frac{x^2-5x}{x^2+8x+16} \\
&= \frac{(x+4)}{(x-5)(x+5)}\cdot\frac{x(x-5)}{(x+4)(x+4)} \\
&= \frac{(x+4)x(x-5)}{(x-5)(x+5)(x+4)(x+4)} \\
&= \frac{\cancel{(x+4)}x\cancel{(x-5)}}{\cancel{(x-5)}(x+5)\cancel{(x+4)}(x+4)} \\
&= \frac{x}{(x+5)(x+4)}
\end{aligned}
$$

Note that we left the answer as $\dfrac{x}{(x+5)(x+4)}$, and it can also be multiplied out as $\dfrac{x}{x^2+9x+20}$.

Division

Rational expressions are divided the same way fractions of integers are done, invert the divsort then multiply.

$$\frac{a}{b}\div\frac{c}{d}=\frac{a}{b}\cdot\frac{d}{c}=\frac{ad}{bc}$$

Example Divide: $\dfrac{3x^4}{x^2-2x-3}\div\dfrac{6xy^2}{x^2-9}$

$$
\begin{aligned}
&= \frac{3x^4}{x^2-2x-3}\div\frac{6xy^2}{x^2-9} \\
&= \frac{3x^4}{x^2-2x-3}\cdot\frac{x^2-9}{36xy^2} \\
&= \frac{3x^4}{(x-3)(x+1)}\cdot\frac{(x-3)(x+3)}{6xy^2} \\
&= \frac{3x^4(x-3)(x+3)}{(x-3)(x+1)6xy^2} \\
&= \frac{3x^4(x-3)(x+3)}{2\cdot 3xy^2(x-3)(x+1)} \\
&= \frac{\cancel{3}x^{\cancel{4}\,3}\cancel{(x-3)}(x+3)}{2\cdot\cancel{3}xy^2\cancel{(x-3)}(x+1)} \\
&= \frac{x^3(x+3)}{2y^2(x+1)}
\end{aligned}
$$

The final answer here $\dfrac{x^3(x+3)}{2y^2(x+1)}$ is not multiplied out, and this is fine. It is completely reduced.

Exercises

Simplify the following.

1. $\dfrac{45}{12}$
2. $\dfrac{x-7}{14-2x}$
3. $\dfrac{x^2-2x}{x^2-4}$
4. $\dfrac{3x^3-12x}{6x^3-24x^2+24x}$
5. $\dfrac{x^2+5x+4}{x^3y^2}\cdot\dfrac{x^2y^3}{x^2+2x+1}$
6. $\dfrac{x^2-2x-8}{x^2+7x+10}\div\dfrac{x^2-11x+28}{x^2-x-42}$
7. $\dfrac{3x^2y}{12xy}$
8. $\dfrac{x^2+4x-5}{x^2-1}$
9. $\dfrac{x^3y^4}{x^2-4x+4}\cdot\dfrac{x^2-x-2}{x^6y^4}$
10. $\dfrac{x^2+3x+2}{x^2+5x+4}\div\dfrac{x^2-x-6}{x^2+2x-15}$
11. $\dfrac{cx+x^2}{ca^2+a^2x}$
12. $\dfrac{x^3-b^2x}{x^2+2bx+b^2}$
13. $\dfrac{x^4-b^4}{x^5-b^2x^3}$
14. $\dfrac{x^2-y^2}{x^4-y^4}$
15. $\dfrac{a^4-x^4}{a^3-a^2x-ax^2+x^3}$