0.1 Special Factoring

The difference of squares and perfect square forms can be factored using previous techniques for polynomials of the form ax^2+bc+c. However, differences of squares and perfect squares occur so often that they are worth special attention and practice. In fact, these perfect square and difference of squares forms seem to occur more often and more naturally than everything else that could possibly call for some factoring to be done.

Difference of Squares

$$a^2 - b^2 = (a-b)(a+b)$$

Anytime you see a difference of squares, it factors.

Example Factor $x^2 - 25$.

$25 = 5^2$, so we can make the difference of squares explicit: $x^2 - 5^2$. Then, with a=x and b=3 we have

$$\begin{aligned} x^2 - 25 &= x^2 - 5^2 \\ &= (x+5)(x-5) \end{aligned}$$

Perfect Squares

$$\begin{aligned} a^2 + 2ab + b^2 &= (a+b)^2 \\ a^2 - 2ab + b^2 &= (a-b)^2 \end{aligned}$$

To check these, multiply out $(a+b)^2$and $(a-b)^2$ to see what happens.

$$\begin{aligned} (a+b)^2 &= (a+b)(a+b) \\ &= a^2 + ab + ab + b^2 \\ &= a^2 + 2ab + b^2 \\ (a-b)^2 &= (a-b)(a-b) \\ &= a^2 - ab - ab + b^2 \\ &= a^2 - 2ab + b^2 \end{aligned}$$

Examples Factor the following.

1. $x^2 + 10x + 25$
   You should be clued in by the fact that $x^2$is a perfect square and that $25 = 5^2$, another perfect square. Looking at the form, we see that $x^2 + 10x + 25 = (x+5)(x+5) = (x+5)^2$.
2. $4x^2 - 12xy + 9y^2$
   We see that $4x^2 = (2x)^2$ and that $9y^2 = (3y)^2(a = 2x, b = 3y)$, and $2ab = 2(2x)(3y) = 24xy$. All fits, and the negative in -12xy tells us to look at the $(a-b)^2$form. Thus, we have $4x^2 - 12xy + 9y^2 = (2x-3y)^2$.
3. $16x^2 - 12x + 9$
   Well, $16x^2 = (4x)^2$ and $9 = 3^2$ ($a = 4x, b = 3$), but $2ab = 2(4x)(3) = 24x$. Thus, we do not have a perfect square form to work with. This failure to match a perfect square form does not mean that $16x^2 - 12x + 9$ does not factor. It just is not a perfect square.

Sum and Difference of Cubes

There is no easy technique to factor cubic (3d degree) polynomials in general. Some are quite difficult and posed challenges for the most experienced mathematicians during Renaisance times–during which formulas were developed to solve general cubic equations. However, some cubic polynomials are easy to factor, and so we mention some of these.

$$\begin{aligned} a^3 + b^3 &= (a+b)(a^2 - ab + b^2) \\ a^3 - b^3 &= (a-b)(a^2 + ab + b^2) \end{aligned}$$

The key to using these formulas–as with any formula–is to carefully establish the replacement for a letter in the formula.

Example Factor $x^3 - 27$

We recognize that $27 = 3^3$, so we have a difference of perfect cubes, $x^3 - 3^3$, with a=x and b=3. Carefully replace a with x and replace b with 3 to obtain

$$\begin{aligned} x^3 - 3^3 &= (x-3)(x^2 + x3 + 3^2) \\ &= (x-3)(x^2 + 3x + 9) \end{aligned}$$

Example Factor $8x^3 + 125y^3$

We recognize that $8x^3 = (2x)^3$and that $125y^3 = (5y)^3$, so our sum of cubes is $(2x)^3 + (5y)^3$ with a=2x and b=5y. On replacement we obtain

$$\begin{aligned} &= (2x)^3 + (5y)^3 \\ &= (2x+5y)((2x)^2 - 2x \cdot 5y + (5y)^2) \\ &= (2x+5y)(4x^2 - 10xy + 25y^2) \end{aligned}$$

The factor $(4x^2 - 10xy + 25y^2)$ looks like it might fit one of our perfect square forms. Alas, no, it does not factor further. In fact, it is hopeless to factor the $(a^2 + ab + b^2)$ or $(a^2 - ab + b^2)$ terms from these formulas if the polynomial was only a cubic.

Exercises

Factor.

1. $x^2 - 144$
2. $x^2 + 22x + 121$
3. $9x^2 - 24xy + 16y^2$
4. $27x^3 - 1$
5. $x^3 + 64y^3$
6. $m^2 - n^2$
7. $4a^2 - 9x^2$
8. $16m^2 - 81n^2$
9. $x^4 - 16y^4$
10. $x^3 + 1$
11. $27 + x^3y^3$
12. $64x^6 - y^3$
13. $1 + (x - y)^3$
14. $8 - (2x + 3)^3$
15. $8x^3 - 27y^3$
16. $x^7y - x^4y^4$
17. $x^2 - 10x + 25$
18. $a^2 + 22ab + 121b^2$
19. $9x^2 + 24x + 16$
20. $4y^2 - 36y + 81$
21. $9 - 12x^2 + 4x^2$
22. $x^2 + 2xy + y^2$
23. $4x^2 - 4x + 1$
24. $x^4 + 6x^2y + 9y^2$
25. $9x^4 + 12x^2 + 4$