0.1 Factoring $ax^2 + bx + c$

There are a couple good ways to factor these polynomials completely and surely, every time. Trial and error is not a good method. When we "factor" polynomials, we mean factoring over the integers. There is no point in learning techniques to factor with fractions or irrational numbers. The following method is called Factoring by Grouping, and it always works.

Theorem Consider a polynomial ax^2+bx+c. If you can find two integers p and q so that
1) $pq = ac$
2) $p + q = b$
Then the polynomial $ax^2 + bx + c = (ax^2+px)+(qx+c)$ factors by grouping.

If such p and q cannot be found, then $ax^2 + bx + c$ is already factored completely.

Example Factor $9x^2 - 6x - 8$.
Here $ac = 9(-8) = -72$, and b = -6.
We write down the pairs with a product of 72 (and worry about adjusting the sign later.)

1	72
2	36
3	24
4	18
6	12
8	9

The product is -72, so one number from each pair must be negative, and the sum is -6, so we must make the larger number from each pair negative.

1	-72
2	-36
3	-24
4	-18
6	-12
8	-9

In scanning, we see that the pair {6,-12} adds up to be -6. At this discovery, we know that the polynomial factors.

Rewrite the polynomial replacing b with (p + q), in this case replacing -6 with (6 - 12), then expand using the distributive law.

$$\begin{aligned} 9x^2 - 6x - 8 &= 9x^2 + (6-12)x - 8 \\ &= 9x^2 + 6x - 12x - 8 \\ &= (9x^2 + 6x) + (-12x - 8) \end{aligned}$$

We have grouped the first two terms together and the last two terms together. Now factor each group by factoring out the greatest common factor.

$$3x(3x+2) - 4(3x+2)$$

The common factor for each group is now the binomial $(3x + 2)$, so factor this binomial out.

$$(3x+2)(3x-4)$$

Done! This result should be checked by multiplying out.

$$\begin{aligned} (3x+2)(3x-4) &= 3x \cdot 3x - 3x \cdot 4 + 2 \cdot 3x - 2 \cdot 4 \\ &= 9x^2 - 12x + 6x - 8 \\ &= 9x^2 - 6x - 8 \checkmark \end{aligned}$$

Factoring of polynomials is an important skill for both practical and theoretical reasons, but it is far more important to know how to factor polynomials–surely with confidence–than it is to develop high speed. In practice, you will usually come across fairly simple polynomials when factoring is a necessary skill. As to why factoring of polynomials is considered unreasonably difficult by some, most blame rests on use of trial and error methods which lead to frustration. In fact, some polynomials do not factor further over the integers, yet the method above of factoring by grouping will always let you know when further factoring is impossible.

Example Factor $12x^2 - 19x - 14$ completely.
First, we write down every pair of integers with a product of -168. We need one of them to be negative because the product -168 is negative, and we make the larger number of each pair negative because the sum needs to be negative, -14.

$$\begin{array}{rcl} (12)(-14) & = & -168 \\ 1 & & -168 \\ 2 & & -84 \\ 3 & & -56 \\ 4 & & -42 \\ 6 & & -28 \end{array}$$

7	-24
8	-21
12	-14

None of these pairs add up to -14, so we confidently conclude that $12x^2 - 19x - 14$ is already factored completely.

Exercises

Factor the following completely.

1. $5x^2 + 6x + 1$
2. $2t^2 + 5t - 12$
3. $9x^2 + 3x + 2$
4. $4z^2 + 5z - 6$
5. $30y^2 - 87y + 30$
6. $15b^2 - 43b + 22$
7. $360y^2 + 4y - 4$
8. $3x^2y - 6xy - 45y$
9. $20x^2 - 5$
10. $4x^2 - 12x + 9$
11. $x^2 + 7x + 12$
12. $x^4 - 4x^3 + 3x^2$
13. $6x^2 - 11x - 10$
14. $x^2 + 14x + 49$
15. $3x^2 + 20x + 12$
16. $14x^2 + 5x - 1$
17. $8x^2 - 14x - 15$
18. $20a^2 - 27a + 9$
19. $16x^2 + 16x + 3$
20. $15y^2 + 4y - 4$
21. $22x^2 - 19x + 4$
22. $30x^2 + 41x + 6$
23. $10x^2 - 3x - 18$
24. $30x^2 + 17x - 2$
25. $36x^2 - 19x - 6$
26. $49a^2 - 42a + 8$
27. $50y^2 - 55y + 14$