0.1 Factoring $x^2 + bx + c$

$$(x+p)(x+q) = x^2 + (p+q)x + pq$$

In words, if $x^2 + bx + c$ can be factored, there are two integers p and q so that

1. $pq = c$
2. $p + q = b$

There are infinitely many pairs of integers that add up to be b, whatever b is, but the number of pairs of integers with a product of c is finite and typically small. Thus, the technique is to write down all the positive pairs of integers whose product is b, adjust the signs (+ or -) for each pair, then scan to see if a pair adds up to c.

Example Factor $x^2 - 5x - 6$.
Here c = -6 and b = -5.
The pairs of integers with a product of 6 are {1,6} and {2,3}. The order does not matter, and we adjust the signs of the numbers next.

One of the numbers in each pair nust be negative because c is negative. Further, because b is negative, we must make the larger number in each pair negative. We now have adjusted pairs {1,-6} and {2,-3}.

The pair {1,-6} works because 1+(-6)=-5, so we know that we can factor $x^2 - 5x - 6$ with the pair: $(x+1)(x-6)$.

Check: $(x+1)(x-6) = x^2 - 6x + x - 6 = x^2 - 5x - 6$. $\surd$

Example Factor $x^2 + 4x + 8$.
Here c = 8 and b = 4.
The pairs of integers with a product of 8 are {1,8} and {2,4}.

Both numbers in the pairs must be negative or both positive since their product c must be positive 8. Further, because their sum b is positive 4, both numbers in each pair must be positive. We now have adjusted pairs {1,8} and {2,4}.

Neither pair works because neither sum is 4. Thus, $x^2 + 4x + 8$ is already completely factored.

This last example of a polynomial which does not factor further is not a failure! The goal is to factor polynomials completely over the integers, and this has been done.

Exercises

Factor the following completely.

1. $x^2 + 7x + 6$
2. $x^2 - 5x - 6$
3. $x^2 - 49$
4. $y^2 - 6y + 9$
5. $5x^2 - 5x - 30$
6. $4x^2y + 20xy - 56y$
7. $2x^4 - 2x^3 - 12x^2$
8. $x^2 - 4x - 33$
9. $x^2 + 5x - 6$
10. $x^2 - 36$
11. $x^2 - 12x + 20$
12. $x^2 - y^2$
13. $m^2 - n^2$
14. $a^2 + 3a + 2$
15. $x^2 + 9x + 18$
16. $x^2 - 5x + 6$
17. $a^2 - 7a + 10$
18. $y^2 - 10y + 16$
19. $c^2 - c - 6$
20. $x^2 + 4x - 5$
21. $x^3 + 5x^2 - 6x$
22. $y^2 + 8y - 65$
23. $a^2 - 4a - 77$
24. $x^2 - 2x - 63$
25. $a^2 + 10a - 75$
26. $a^2 - 24a + 143$
27. $30 + 11x + x^2$
28. $21 + 10a + a^2$
29. $35 - 12x + x^2$
30. $36 - 13x + x^2$
31. $c^2 + 2cd - 3d^2$
32. $a^2 + 8ax + 15x^2$
33. $x^2 - xy - 20y^2$