1 Factoring

Factoring a polynomial rests on factoring integers like, say, 21. We know that factors of 21 are 7 and 3 because we can recall that $3 \cdot 7 = 21$, and we know that 3 and 7 do not have any integer divisors except for 1 and themselves. Thus, 21 completely factored is $3 \cdot 7$. Factoring polynomials such as $x^2 + 4x$ requires technique beyond multiplication facts, yet we can recognize a factored polynomial like $x(x+4)$ just as we recognize that $3 \cdot 7 = 21$.

In fact, we only attempt to factor polynomials with integer coefficients. Further, we focus on common types of factoring problems.

1.1 Greatest Common Factor

The object here is similar to finding the least common denominator among fractions or the least common multiple of some integers. We want to find the largest common factor of the terms in a polynomial, quite similar.

First consider two expressions: $4x, 12x^2$
We can factor both of these.

$$\begin{aligned} 4x &= 2 \cdot 2 \cdot x \\ 12x^2 &= 2 \cdot 2 \cdot 3 \cdot x \cdot x \end{aligned}$$

Looking at these two expressions, we want to find the largest factor which is entirely contained in both. We see two 2s in common, and one x, and that is it. Thus the greatest common factor is $2 \cdot 2 \cdot x = 4x$. We do essentially this work in factoring out athe greatest common factor of a polynomial.

Example Factor out the greatest common factor of the polynomial $4x + 12x^3$
We saw already that $4x$ was the greatest common factor for the expressions $4x$ and $12x^2$, so we factor it out leaving whatever is left. Note that when 4x is factored out of 4x, there is a 1 left, and when 4x is factored out of $12x^2$, there is 3x left.

$$\begin{aligned} 4x + 12x^2 &= 4x \cdot 1 + 4x \cdot 3x \\ &= 4x(1 + 3x) \end{aligned}$$

The way to make sure that factoring is correct is to check, multiply out using the distributive law, and see that the original polynomial results.

$$\begin{aligned} 4x(1 + 3x) &= 4x \cdot 1 + 4x \cdot 3x \\ &= 4x + 12x^2 \ \surd \end{aligned}$$

Exercises

Factor out the greatest common factor.

1. $3x^2 + 5x^5$
2. $5a^2 - 25$
3. $16 + 64xy$
4. $2a - 2a^2$
5. $15a^2 - 225a^4$
6. $x^3 - x^2$
7. $a^2 + ab$
8. $6a^3 + 2a^4 + 4a^5$
9. $7x - 7x^3 + 14x^4$
10. $3x^3 - x^2 + x$
11. $a^3 - a^2y + ay^2$
12. $3a(x+y) + 5mb(x+y) - 9d^2x(x+y)$
13. $4(a-b) - 15xy(a-b) + (a-b) - 5a^2b(a-b)$
14. $4x^3y - 12ax^2 - 8xy^3$
15. $6ax^3y^5 - 4ax^2y^6 + 2axy^7 - 2a^2xy^9$
16. $51x^5y - 34x^4y^2 + 17x^2y^4$
17. $6a^2b^2 - 3a^3b^3c - 9ab^3c + 3abc^2$
18. $3ax^7 - 24ax + 9ax^5 - 3ax^4 - 9ax^6$
19. $7x^2 - 20x$
20. $x^3y - 3x^2y^2 + 7xy^3$

Factor by grouping.

21. x(a + 2) + 7(a + 2)
22. $x^2 + 2x + 2xy + 4y$

23. $ax + ay + bx + by$
24. $x^2 + ax + bx + ab$
25. $ax^2 + ay^2 - bx^2 - by^2$
26. $x^2 - ax + 5x - 5a$
27. $x^2 + mxy - 4xy - 4my^2$
28. $2x^4 - x^3 + 4x - 2$
29. $mx - ma - nx + na$
30. $x^3 + x^2 + x + 1$
31. $y^3 - y^2 + y - 1$
32. $3ax + 3ay - 2bx - 2by$
33. $6amx + 3amy - 6anx - 3any$
34. $x^2 - 2x - 3xy + 6y$

Multiply out.

35. (x - 2)(x + 5)
36. (x - 5)(x + 5)
37. (x + 4)(x + 5)(x + 6)