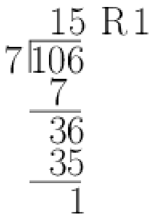
0.1 Division of Polynomials

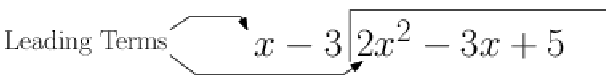
Just as multiplication of polynomials is similar to ordinary multiplication of integers, division of polynomials is similar to long division of integers with a remainder but without borrowing. Recall the process of dividing 106 by 7.



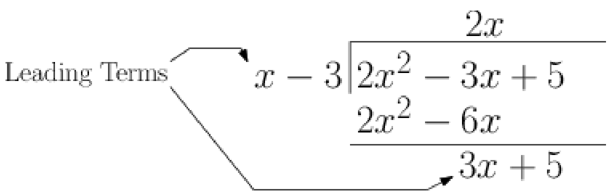
In long division, one does not have to know how many times the divisor goes into the dividend (dividend = thing being divided). One can look at a digit or group of digits at a time, repeatedly subtracting, until the remiander is less than the divisor.

Polynomial division is actually quite a bit more straightforward because there is no guessing and no borrowing during subtraction.

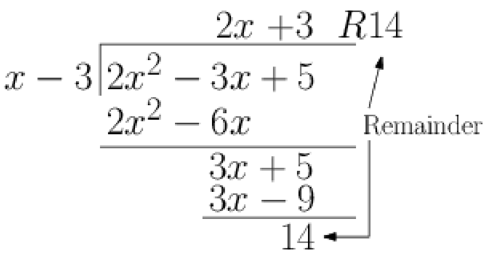
Consider the problem $(2x^2 - 3x + 5) \div (x - 3)$, or equivalently $\dfrac{2x^2 - 3x + 5}{x - 3}$.



The ratio of the leading terms is $\dfrac{2x^2}{x} = 2x$. Put the $2x$ in the correct column for the quotient, and subtract $2x(x - 3) = 2x^2 - 6x$.



The ratio of the leading terms is now $\dfrac{3x}{x} = 3$. Put 3 in the correct quotient counn, and subtract $3(x - 3) = 3x - 9$.



There are two ways to write the results from division. We already know that $\dfrac{Dividend}{Divisor} = Quotient + \dfrac{Remainder}{Divisor}$. This is the usual way to write the result of division and in $\dfrac{106}{7} = 15 + \dfrac{1}{7}$. This method can be used for the results of polynomial division too: $\dfrac{2x^2 - 3x + 5}{x - 3} = 2x + 3 + \dfrac{14}{x - 3}$.

Another method for expressing the result from division is also useful, and it is called The Division Algorithm. It is sometimes handy to label the dividend polynomial as p(x), the divisor as d(x), the quotient as q(x), and the remainder as r(x).

$$\begin{aligned} Dividend &= Quotient \cdot Divisor + Remainder \\ p(x) &= q(x) \cdot d(x) + r(x) \end{aligned}$$

In this form, $106 = 15 \cdot 7 + 1$, and $2x^2 - 3x + 5 = (2x + 3)(x - 3) + 14$.

Synthetic Division

In the above problem, there were redundant steps in carrying out $(2x^2 - 3x + 5) \div (x - 3)$. The leading term in the dividend is always eliminated, and there is always a single subtraction. These steps can be condensed into a matter of multiplying and adding in this fashion.

Consider a polynomial $\left(a_n x^n + a_{n-1} x^{n-1} + \cdots + a_2 x^2 + a_1 x + a_0\right) \div (x - r)$. Write down the coefficients in this fashion.

r	a_n	a_{n-1}	$\cdots$	a_2	a_1	a_0
	$\downarrow$	$r \cdot a_n$				
	a_n	$a_{n-1} - r \cdot a_n$				*Remainder*

The process is started by carrying the leading coefficient $a_n$to the bottom. Then, multiply $r \cdot a_n$and put this product in the second row below the next term and subtract putting the result in the bottom row. Continue this process until there is a number in the bottom row to the far right–which is the remainder. The other numbers in the second row form the coefficients of the quotient. This process is probably easier to show than to describe generally.

Example Divide $(2x^2 - 3x + 5) \div (x - 3)$ using synthetic division.

$$\begin{array}{r|rrr} 3 & 2 & -3 & 6 \\ & & & \\ \hline & & & \end{array} \qquad \begin{array}{r|rrr} 3 & 2 & -3 & 5 \\ & \downarrow & & \\ \hline & 2 & & \end{array}$$

$$\begin{array}{r|rrr} 3 & 2 & -3 & 5 \\ & & 6 & \\ \hline & 2 & & \end{array} \qquad \begin{array}{r|rrr} 3 & 2 & -3 & 5 \\ & & 6 & \\ \hline & 2 & 3 & \end{array}$$

$$\begin{array}{r|rrr} 3 & 2 & -3 & 5 \\ & & 6 & 9 \\ \hline & 2 & 3 & \end{array} \qquad \begin{array}{r|rrr} 3 & 2 & -3 & 5 \\ & & 6 & 9 \\ \hline & 2 & 3 & [14] \end{array}$$

The number in the brackets, 14, is the remainder. The quotient is one degree less than the dividend $2x^2 - 3x + 5$, and its coefficients are 2 and 3, so the quotient is $2x + 3$. The answer can be expressed in two ways.

$$\begin{aligned} \frac{2x^2 - 3x + 5}{x - 3} &= 2x + 3 + \frac{14}{x - 3} \\ 2x^2 - 3x + 5 &= (x - 3)(2x + 3) + 14 \end{aligned}$$

Exercises

Divide (with remainder) by hand showing your steps. Put your answer in the form
$dividend = quotient + \dfrac{remainder}{divisor}$

1. $315 \div 23$
2. $(5x^2 - 4x + 9) \div (x + 2)$
3. $(x^3 + x^2 - 2x + 3) \div (x + 1)$
4. $\dfrac{x^3 + 5}{x - 2}$
5. $\dfrac{x^3 + 2x^2 - 7x + 8}{x - 1}$

Divide using synthetic division. Put your answers in the form
$dividend = quotient + \dfrac{remainder}{divisor}$

6. $\left(x^3 + 3x^2 - 4x + 5\right) \div (x - 1)$
7. $\left(2x^3 + 4x^2 - x - 5\right) \div (x + 2)$
8. $\left(3x^5 - x^4 + x^3 - x^2 + x + 3\right) \div (x + 1)$
9. $\left(2x^4 - x^2 + 2\right) \div (x - 2)$