5.4 Applications of Linear Systems

There are a great many applications which arise which boil down to solving a system of linear equations. Some methods of solving linear systems are more useful than others.

1. Substitution always works, but it is usually tedious for decimal coefficients (as in a linear equations like $2.3x - 7.1y = 93.5$) and ends up being more difficult than elimination if the answers are fractions. Substitution is fine when the coefficients are integers, and it is easy to solve for one variable in terms of another.
2. Elimination is nice when the coefficients are integers.
3. Cramer's rule is straightforward and certain (when a solution exists) for decimal coefficients. Cramer's is convenient for 2x2 systems, useful for 3x3 systems, but usually tedious for systems larger than 3x3.
4. For 3x3 and larger systems, calculators can be very useful. We are not covering row-reduction and other related methods which are the most efficient for larger systems of linear equations.

The trick in solving a word problem is to express the problem as a system of equations which, on solution, delivers the solutions.

We focus here on solving 2x2 systems: two equations in two unknowns. The first thing to do is select variables for the unknowns, then extract the equations from the words. Two equations are needed for two unknowns.

Example There are 34 quarters and nickels for a total of \$6.30 . How many quarters and nickels are there?

Choose variables which are the requested unknowns.
x = number of quarters
y = number of nickels
Now, extract equations in these two variables from the problem statement.
1) There are 34 coins, so x + y = 34
2) The total value of quarters and nickels is \$6.30, each quarter is worth .25, and each nickel is worth .05, so
.25x + .05y = 6.30

Here is the linear system to solve:

$$\left\{ \begin{array}{rcl} x + y & = & 34 \\ .25x + .05y & = & 6.30 \end{array} \right\}$$

We will use Cramer's rule here, and solve for x first:

$$\begin{aligned} x &= \frac{\begin{vmatrix} 34 & 1 \\ 6.3 & .05 \end{vmatrix}}{\begin{vmatrix} 1 & 1 \\ .25 & .05 \end{vmatrix}} = \frac{(34)(.05) - (1)(6.3)}{(1)(.05) - (1)(.25)} \\ &= \frac{-4.6}{-.2} = 23 \end{aligned}$$

We could solve for y using Cramer's rule, but note that the first equation is simple, so we can more easily use it to substitute x in and find a solution for y.

$$\begin{aligned} x + y &= 34 \\ 23 + y &= 34 \\ y &= 34 - 23 = 11 \end{aligned}$$

Thus, we have 23 quarters and 11 nickels.

Example \$10,000 was invested in stock and bonds. The bonds earn a sure 4% (.04), and the stock is projected to earn 7% (.07) per year. If a conservative by adequate investment in stock and bonds is designed to earn 6.2% (.062), how much is invested each in stock and bonds?
Let x = amount in stock
Let y = amount in bonds
Then x + y = 10000, and
.07x + .04y = .062(10000)
The second equation can be simplified to
.07x + .04y = 620
We have the system

$$\left\{ \begin{array}{rcl} x + y & = & 10000 \\ .07x + .04y & = & 620 \end{array} \right\}$$

Solving by Cramer's rule looks like it would work well.

$$\begin{aligned} x &= \frac{\begin{vmatrix} 10000 & 1 \\ 620 & .04 \end{vmatrix}}{\begin{vmatrix} 1 & 1 \\ .07 & .04 \end{vmatrix}} \\ &= \frac{(10000)(.04) - (1)(620)}{(1)(.04) - (1)(.07)} \\ &= \frac{-220}{-.03} = 7333.33 \end{aligned}$$

It looks quick to use the first equation to solve for y using the found value of x.

$$\begin{aligned} x + y &= 10000 \\ 7333.33 + y &= 10000 \\ y &= 10000 - 7333.33 \\ &= 2666.67 \end{aligned}$$

Thus, we invest \$7,333.33 in stock and \$2,666,67 in bonds.

Exercises

Solve each application problem. First, select two variable for the two unknowns. then write a 2x2 linear system, solve the linear system, and express your answer in plain English.

1. There are 48 nickels and quarters which total \$5.80 . How many of each coin is there?
2. A 26% alcohol snapps is mixed with a 44% alcohol brandy. 12 quarts of 31% alcohol punch is needed. How much snapps and how much brandy should be used?
3. A car radiator holds 8 quarters of coolant (a micture of water and anti-freeze), and it should be 65% anti-freeze. There is a new container of 80% antifreeze and an old container of 53% anti-freeze available. How much of each, fluid from the old and new containers, should be used?
4. Chicken feed should contain 6% protein for optimum growth. The Super Grain brand contains 5.4% protein, and the Big Bird brand contains 7.1% protein. How much of each should be used to make a 100 lb mixture containing 6% protein?
5. A municipal bond earns 4.2% per year, and a money market certificate earns 3.14% per year. For a total investment of \$230,000 , how much should be placed in each type of investment to earn \$8211 per year?
6. A boiler requires a 1.25% TSP solution in water to minimize corrosion, and this boiler always holds 1250 gallons of water solution. The boiler solution only has 1.13% TSP, so a 10% TSP solution must be injected–and an equal amount of current boiler solution must be removed. How much 10% TSP solution must be injected?