5.3 Determinants and Cramer's Rule

Matrices

A matrix is a grid indexed by row and column, and a matrix is usually referred to by a capital letter. For example, consider matrix $A = \begin{bmatrix} 3 & -6 \\ 8 & 1 \end{bmatrix}$. Matrix A has 2 rows and 2 columns, and it is called a 2x2 matrix for its dimension being (# rows)x(#columns). Elelments of matrix A are identified by row then column in subscripts. For the matrix A just defined, $a_{11} = 3$, $a_{12} = -6$,$a_{21} = 8$, and $a_{22} = 1$.

Determinant

The determinant of a 2x2 matrix is a real number. To take the determinant of a matrix A, the matrix is surrounded by vertical bars.

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$$

Example Find the determinant of matrix $A = \begin{bmatrix} 3 & -6 \\ 8 & 1 \end{bmatrix}$.

$$\begin{aligned} \begin{vmatrix} 3 & -6 \\ 8 & 1 \end{vmatrix} &= (3)(1) - (-6)(8) \\ &= 3 - (-48) = 3 + 48 = 51 \end{aligned}$$

The determinant arises as a useful operation in many situations. We will examine one application in solving 2x2 linear systems.
Consider a general 2x2 linear system $\begin{cases} a_1x & +b_1y & = c_1 \\ a_2x & +b_2y & = c_2 \end{cases}$, and solve it by substitution.
Take the first equation and solve for y.

$$\begin{aligned} a_1x + b_1y &= c_1 \\ b_1y &= c_1 - a_1x \\ y &= \frac{c_1 - a_1x}{b_1} \end{aligned}$$

Substitute y in the second equation for $\dfrac{c_1 - a_1x}{b_1}$.

$$\begin{aligned} a_2x + b_2y &= c_2 \\ a_2x + b_2\left(\frac{c_1 - a_1x}{b_1}\right) &= c_2 \\ a_2b_1x + b_2c_1 - a_1b_2x &= b_1c_2 \\ a_2b_1x - a_1b_2x &= b_1c_2 - b_2c_1 \\ x &= \frac{b_1c_2 - b_2c_1}{a_2b_1 - a_1b_2} \\ x &= \frac{b_2c_1 - b_1c_2}{a_1b_2 - a_2b_1} \\ x &= \frac{\begin{vmatrix} c_1 & b_1 \\ c_2 & b_2 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}} \end{aligned}$$

Similary, replacing this value $= \dfrac{b_2c_1 - b_1c_2}{a_1b_2 - a_2b_1}$ for x in either equation yields $y = \dfrac{\begin{vmatrix} a_1 & c_1 \\ a_2 & c_2 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}}$.

This solution for a general 2x2 system in the form of quotients of determinants is called Cramer's Rule. The determinant in the denominator for each solution of x and y is the same, and it is called the determiant of the coefficient matrix. the coefficient matric is obtained from forming a square matrix from the coefficients of the two equations.

One cannot divide by zero, and so if $\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} = 0$, there can be no solution as a point. This leave two possibilities: no solution or infinitely many solutions on a line. Cramer's rule does not say which is the case if the determinant of the coefficient matrix is zero. However, Cramer's rule is especially handy in the typical situation where there is a unique solution. One can usually look at a 2x2 system and tell if there is going to be a problem just by calculating if $a_1b_2 = a_2b_1$; if so, the determinant of the coeficient matrix is zero.

Example Solve the system using Cramer's rule. $\begin{cases} 4x + 7y = 5 \\ 3x - y = -4 \end{cases}$

$$x = \frac{\begin{vmatrix} 5 & 7 \\ -4 & -1 \end{vmatrix}}{\begin{vmatrix} 4 & 7 \\ 3 & -1 \end{vmatrix}}$$

$$= \frac{5(-1)-7(-4)}{4(-1)-7(3)}$$
$$= \frac{-5+28}{-4-21} = \frac{23}{-25} = -\frac{23}{25}$$

Note that we should really start by taking the determinant in the denominator since this, if zero, will prevent us from wasting effort taking the determinant in the numerator. In any case, we have $x = -\frac{23}{25}$. Note that the determinant in the denominator (the determinant of the coefficient matrix) was -25. We can use this value in solving for y.

$$y = \frac{\begin{vmatrix} 4 & 5 \\ 3 & -4 \end{vmatrix}}{-25} = \frac{4(-4)-5(3)}{-25}$$
$$= \frac{-16-15}{-25} = \frac{-31}{-25} = \frac{31}{25}$$

Thus, the solution is $\left(-\frac{23}{25}, \frac{31}{25}\right)$.

Cramer's Rule

$$\begin{cases} a_1x & +b_1y & = c_1 \\ a_2x & +b_2y & = c_2 \end{cases} \qquad \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$$

$$x = \frac{\begin{vmatrix} c_1 & b_1 \\ c_2 & b_2 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}} \qquad y = \frac{\begin{vmatrix} a_1 & c_1 \\ a_2 & c_2 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}}$$

Exercises

Calculate the determinants.

1. $\begin{vmatrix} 3 & 4 \\ 2 & 5 \end{vmatrix}$
2. $\begin{vmatrix} 6 & -4 \\ 8 & -9 \end{vmatrix}$
3. $\begin{vmatrix} 5 & -12 \\ -10 & 24 \end{vmatrix}$
4. $\begin{vmatrix} 1.03 & -4.68 \\ -7.03 & 11.5 \end{vmatrix}$

Use Cramer's Rule to solve each system, if possible.

5. $\begin{cases} 3x & +5y & = -2 \\ x & +8y & = 4 \end{cases}$
6. $\begin{cases} 4x & -6y & = 9 \\ 7x & +2y & = 10 \end{cases}$
7. $\begin{cases} 3x - 5y = 2 \\ -6x + 10y = 4 \end{cases}$
8. $\begin{cases} 3.1x & +5.4y & = -4.2 \\ 8.2x & +1.4y & = 9.1 \end{cases}$
9. $\begin{cases} 4x & -10y & = 11 \\ x & +7y & = 2 \end{cases}$
10. $\begin{cases} 43.2x + 71.9y = 104.5 \\ 11.6x - 17.2y = 31.1 \end{cases}$