1 Systems of Linear equations and inequalities

Solving linear systems of equations and inequalities (linear systems) is a huge portion of the mathematical work needed in applications. Also, solving linear systems is much more important now than 60 years ago because computers perform arithmetic quickly and accurately. It is thus natural to solve problems by turning their solutions into a matter of solving linear systems.

Somewhat ironically (yet sensibly), we focus here on solving rather small systems of linear equations without using calculators or computers, and these will be 2x2 and 3x3 linear systems. A 2x2 linear system is a set of two equations in two unknowns, and a 3x3 system is a set of three equations in three unknowns. 2x2 linear systems can be visualized on the plane, and 3x3 linear systems can be visualized in 3-space or simply “space”. The good news is that there are a great many practical uses for relatively small 2x2 and 3x3 systems. The further good news is that many methods for solving 2x2 and 3x3 systems apply directly to solving larger linear systems, say 100x100 and even (billion)x(billion) linear systems.

You may already know that hand graphing calculators can quickly and easily solve small linear systems up to their limits, say up to 10x10 linear systems. This is nice, yet we are less interested in cranking out numbers to satisfy 2x2 and 3x3 systems than we are in figuring out how one obtains solutions and why solutions may be impossible.

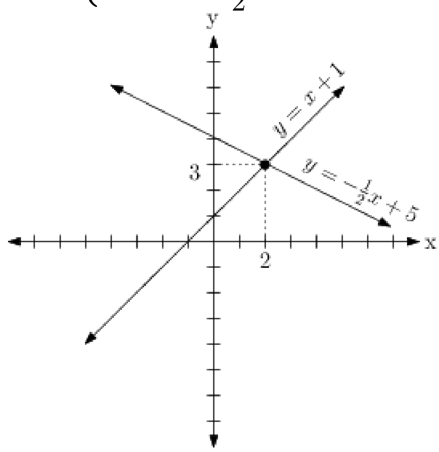
In fact, there are some who have decided that it is useless to know anything about the details of solving linear systems because it is impractical for humans to solve large systems except by using a computer. With the same facts, I take precisely the opposite view which is that it is essential to know the details because it is increasingly attractive to solve large systems by using a computer.

5.1 Graphing Linear Systems, and Solutions by Substitution

An equation in two variable, say x and y, describes a line on the plane. Two equations in two variables (a 2x2 linear system) describe two lines on the plane. Three things can happen with two lines on the plane.

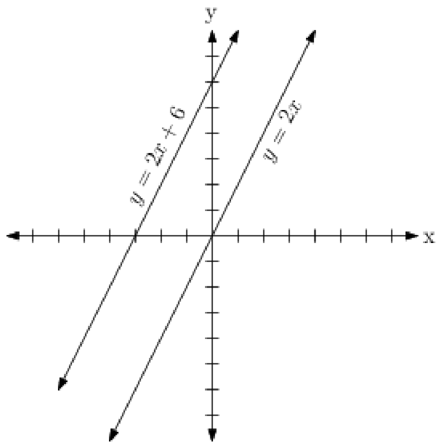
Case 1 There is one solution point where the two lines intersect. The solution point is (2,3) where x = 2 and y = 3, and it is not hard to find with careful graphing.

Example $\begin{cases} y = x+1 \\ y = -\frac{1}{2}x+4 \end{cases}$.



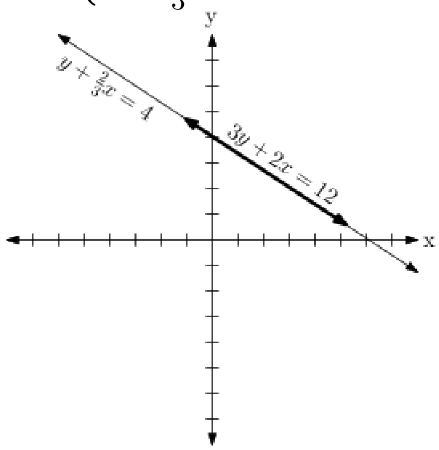
Case 2 The two lines are parallel and do not touch. The slopes are clearly the same, but the lines are not exactly the same, so they are parallel..

Example $\begin{cases} y = 2x+6 \\ y = 2x \end{cases}$



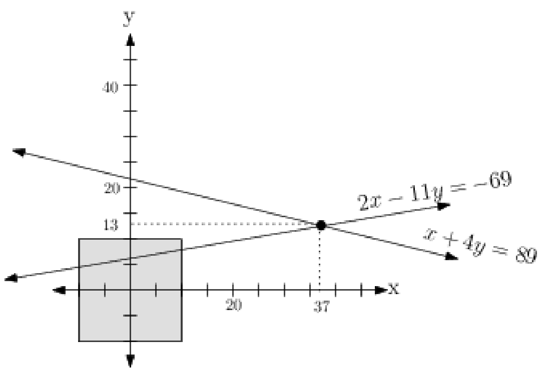
Case 3 The two equations are equivalent. Multiply the second line on both sides by 3, and the resulting equation is the same as the first.

Example $\begin{cases} 3y + 2x &= 12 \\ y + \frac{2}{3}x &= 4 \end{cases}$



Graphing linear systems is a fine way to see what is going on, yet it is a poor general method. By "general method", we mean a method which applies to every situation or, at least, a rather large class of situations. The ability to find a solution by graphing depends on the accuracy of the graph itself.

Consider the linear system $\begin{cases} 2x - 11y &= -69 \\ x + 4y &= 89 \end{cases}$



The intersection point is (37, 13), and this is going to be very difficult to hit without very careful and laborous hand graphing. Finding this intersection point isn't straightforward with a graphing calculator either as the default graphing window (shown gray) is going to miss it along with one line, so finding the solution requires some hunting around with zooming and boxing. Further, one first has to solve for y in each equation in order to get started entering equations into the calculator. This is not to say that graphing is not useful. Rather, graphing does not eliminate a use for algebra.

Substitution

Substitution is perhaps the fundamental algebraic technique, and we enable substitution in solving linear systems by assuming from the start that every equation in the system is true. Since each equation is true at the same time, we can freely substitute a result from one equation into another. This is the strategy.

1. Pick one equation, and solve for one variable in terms of everything else. E.g., to solve for y, you want to end up with something like $y = Everything\,Else$.
2. Replace the variable in the other equation with $Everything\,Else$ found in the first step, and solve for the remaining variable.
3. Substitute the value of the variable found in one original equation to find the value of the remianing variable.

Example Solve the system by substitution: $\begin{cases} y &= x + 3 \\ y &= 2x + 1 \end{cases}$.

We already have $y = x+3$ in the first equation, so replace y in the second equation with $x+3$.

$$\begin{aligned} y &= 2x + 1 \\ (x + 3) &= 2x + 1 \\ 3 &= x + 1 \\ 2 &= x \end{aligned}$$

We now know that x = 2, so pick one equation, say the first, and replace x with 2.

$$\begin{aligned} y &= x + 3 \\ y &= (2) + 3 \\ y &= 5 \end{aligned}$$

Hence, the solution is (2, 5). It never hurts to check solutions either.

Check (2, 5) in $y = x + 3$: $5 \stackrel{?}{=} 2 + 3$ $\checkmark$

Check (2, 5) in $y = 2x + 1$:$5 \stackrel{?}{=} 2(2) + 1$ $\checkmark$

Substitution always works in that you always find the solution set to a 2x2 linear system, yet the solution set can be a point, a line, or an empty set. If you do everything right and end up with a contradiction, say 3 = 1, then the solution set is empty. If you do everything right and end up with an identity, say 5=5, then the solution set is a line.

Example Find the solution set of the system $\begin{cases} 2x - 11y &= -69 \\ x + 4y &= 89 \end{cases}$.
This system was used in a previous example showing solution by graphing, but we solve it here using substitution. It is easiest to solve for x in the second equation.

$$\begin{aligned} x + 4y &= 89 \\ x &= 89 - 4y \end{aligned}$$

Then, replace x in the first equation with 89 - 4y.

$$\begin{aligned} 2(89 - 4y) - 11y &= -69 \\ 178 - 8y - 11y &= -69 \\ 178 - 19y &= -69 \\ -178 + 178 - 19y &= -178 - 69 \\ -19y &= -247 \\ \frac{-19y}{-19} &= \frac{-247}{-19} \\ y &= 13 \end{aligned}$$

Then, take one equation, say the second, and substitute 13 for y and solve for x.

$$\begin{aligned} x + 4y &= 89 \\ x + 4(13) &= 89 \\ x + 52 &= 89 \\ x + 52 - 52 &= 89 - 52 \\ x &= 37 \end{aligned}$$

Thus, the solution is $(37, 13)$, and the solution set is $\{(37, 13)\}$.

Example Find the solution set for the system $\begin{cases} 2x - 4y &= 6 \\ -x + 2y &= -3 \end{cases}$.
Solve for x in the second equation.

$$\begin{aligned} -x + 2y &= -3 \\ -x &= -2y - 3 \\ x &= 2y + 3 \end{aligned}$$

Replace x in the first equation with 2y + 3.

$$\begin{aligned} 2(2y + 3) - 4y &= 6 \\ 4y + 6 - 4y &= 6 \\ 6 &= 6 \end{aligned}$$

We have an identity here. For 2x2 systems, when an identity results during substitution, the solution set is the inifinte set of points on the line. Both equations are equivalent, so we can pick either one to express the solution. It is usual to express the solution set of all points on a line in ordered pair form.

Suppose that x were any real number t. Then, using either equation, we solve for y. Here choose the second equation replacing x with t.

$$\begin{aligned} -t + 2y &= -3 \\ t - t + 2y &= t - 3 \\ 2y &= t - 3 \\ y &= \frac{1}{2}(t - 3) \\ y &= \frac{1}{2}t - \frac{3}{2} \end{aligned}$$

Then, the solution set is $\left\{(t, \frac{1}{2}t - \frac{3}{2}) | t \in \mathbb{R}\right\}$. Note that this is a formal way of saying that the solution is the line $y = \frac{1}{2}t - \frac{3}{2}$.

Example Find the solution to the system $\begin{cases} x + y &= 6 \\ 2x + 2y &= -2 \end{cases}$
Solving for x in the first equations, we obtain $x = 6 - y$. We substitute 6 - y for x in the second equation.

$$\begin{aligned} 2(6 - y) + 2y &= -2 \\ 12 - 2y + 2y &= -2 \\ 12 &= -2 \end{aligned}$$

12 does not equal -2, and so we have a contradiction. There is no solution, but we can say this is a positive way by pointing out that the solution set is $\{\} = \emptyset$. The empty set $\emptyset$is the solution set.

Exercises

Solve using addition

1. $\begin{cases} 3x + 5y = 2 \\ -x - 5y = -4 \end{cases}$

2. $\begin{cases} x &= 2y \\ 3x + 7y &= 130 \end{cases}$

3. $\begin{cases} 6x - 3y &= 3 \\ x + y &= 3 \end{cases}$

4. $\begin{cases} 2x &= 3y \\ 9x - 17y &= -7 \end{cases}$

5. $\begin{cases} 3x - 5y &= 23 \\ 7x + y &= -35 \end{cases}$

6. $\begin{cases} 3x + 5y &= 50 \\ x - 7y &= 8 \end{cases}$

7. $\begin{cases} x + 21y &= 2 \\ 2x + 27y &= 19 \end{cases}$

8. $\begin{cases} 5p + 5q &= 2 \\ p + 3q &= 9 \end{cases}$

9. $\begin{cases} x &= 5y + 3 \\ y &= 2x - 24 \end{cases}$

10. $\begin{cases} a - 2b &= 2 \\ 2a - 6b &= 3 \end{cases}$

11. $\begin{cases} x + y &= 7 \\ 2x + 2y &= 14 \end{cases}$