1.2 Rational and Irrational Numbers

Rational numbers are fractions with an integer for the numerator and a non-zero integer for the denominator. $\frac{1}{2}$, $\frac{-5}{3}$, and $\frac{12}{4}$ are rational numbers. Now, $\frac{12}{4} = 3$, so 3 is a rational number. In fact, all integers are rational numbers because any integer x can be written as $\frac{x}{1}$. Representation is not unique. For example, $\frac{12}{4} = \frac{6}{2} = \frac{300}{100} = \frac{-9}{-3} = \frac{3}{1} = 3$. However, it is usual to write rational numbers in reduced form which is formed by canceling all common factors between numerator and denominator.

This method of reducing fractions to lowest terms is called cancelation.

$$\frac{ac}{bc} = \frac{a}{b}$$

Example $\frac{15}{9}$ would be $\frac{5}{3}$ when completely reduced because a 3 can be factored out of both numerator and denominator and cancelled out.

$$\frac{15}{9} = \frac{5 \cdot 3}{3 \cdot 3} = \frac{5}{3}$$

Multiplication

$$\frac{a}{b} \cdot \frac{c}{d} = \frac{ac}{bd}$$

Example Multiply $\left(\frac{6}{5}\right)\left(\frac{7}{9}\right)$.

$$\left(\frac{6}{5}\right)\left(\frac{7}{9}\right) = \frac{6 \cdot 7}{5 \cdot 9} = \frac{42}{45} = \frac{3 \cdot 14}{3 \cdot 15} = \frac{14}{15}$$

Division of Fractions

To divide a number (dividend) by another number (divisor), invert the divisor and multiply.

$$a \div b = a \cdot \frac{1}{b}$$

Example $\frac{2}{3} \div \frac{3}{5} = \frac{2}{3} \cdot \frac{5}{3} = \frac{10}{9}$

Example $\frac{4}{7} \div 5 = \frac{4}{7} \cdot \frac{1}{5} = \frac{4}{35}$

Example $7 \div \frac{-2}{3} = \frac{7}{1} \cdot \frac{-3}{2} = \frac{-21}{2}$

Example $\frac{\frac{7}{2}}{\frac{4}{7}} = \frac{7}{2} \cdot \frac{7}{4} = \frac{49}{8}$

For a number b, the reciprocal of b is $\frac{1}{b}$. Why does inverting the divisor then multiplying work? It is probably easiest to see for a particular case, say by dividing by 1/2. Dividing 5 by 1/2 ($5 \div \frac{1}{2}$) means counting how many pieces of size 1/2 the number 5 can be sliced, and that would be 10. Thus, dividing by 1/2 is equivalent to multiplying by 2 ($5 \cdot 2 = 10$), the reciprocal of 1/2.

Addition and Subtraction

To add or subtract fractions, the denominators must be exactly the same, then add or subtract the numerators.

$$\frac{a}{c} + \frac{b}{c} = \frac{a+b}{c}$$
$$\frac{a}{c} - \frac{b}{c} = \frac{a-b}{c}$$

Example $\frac{3}{7} + \frac{5}{7} = \frac{3+5}{7} = \frac{8}{7}$

Example $\frac{2}{5} - \frac{3}{5} = \frac{2-3}{5} = \frac{-1}{5}$

The entire trick to addition and subtraction of fractions is making the denominators the same first. Fractions can be converted to equivalent fractions by multiplying both numerator and denominator by the same thing.

$$\frac{a}{b} = \frac{ac}{bc}$$

Note that this is just the cancellation property in reverse. To add two fractions $\frac{a}{b}$ and $\frac{c}{d}$, we want to make two equivalent fractions having the same denominator or common denominator. This common denominator is also called a common multiple, that is a number which contains both denominators as factors.

A common multiple of 3 and 5 is 15, and 15 also happens to be the least common multiple of 3 and 5. There are many (infinitely many) multiples of 3 and 5 including 30, 45, 60, and 6000 to name some more. For convenience and sometimes necessity, it is usual to choose the least commmon multiple.

For example, the least common multiple of 6 and 9 is not 54 ($6 \cdot 9 = 54$) but 18. There is a method to finding the least common multiple of two or more integers which involves factoring each integer to a product of primes.

Example Find the least common multiple of 12 and 18.

$$\begin{aligned} 12 &= 2 \cdot 2 \cdot 3 \\ 18 &= 2 \cdot 3 \cdot 3 \end{aligned}$$

We build up the least common multiple (LCM) by picking the largest number of each prime that occurs in any number. There are two 2s in 12, so we need these. There are two 3s in 18, so we need these. This covers all the primes that occur.

Thus, the LCM of 12 and 18 is $2 \cdot 2 \cdot 3 \cdot 3 = 36$.

Example Add. $\frac{5}{12} + \frac{7}{18}$.

The LCM of 12 and 18 is 36, so we build up each fraction to have this LCM as the denominator.

$\frac{5 \cdot 3}{12 \cdot 3} = \frac{15}{36}$, and $\frac{7 \cdot 2}{18 \cdot 2} = \frac{14}{36}$. Then
$\frac{5}{12} + \frac{7}{18} = \frac{15}{36} + \frac{14}{36} = \frac{15+14}{36} = \frac{29}{36}$

Exercises

Perform the arithmetic then simplify the result.

1. $\frac{19}{0}$
2. $\frac{5}{8} + \frac{3}{4} + \frac{1}{2}$
3. $\frac{2}{3} - \frac{1}{2} + \frac{5}{6}$
4. $-\frac{3}{8} \cdot \frac{16}{9}$
5. $\left(\frac{-2}{5}\right) \div \left(\frac{-8}{25}\right)$
6. $\frac{1}{5} + \frac{7}{5}$
7. $\frac{3}{8} - \frac{7}{8}$
8. $\frac{1}{3} \cdot \frac{6}{5}$
9. $\frac{3}{5} \div \frac{9}{10}$
10. $\frac{2}{5} + \frac{3}{10}$
11. $\frac{5}{4} - \frac{1}{3}$
12. $240 \div 15$
13. $\frac{3}{5} - \frac{1}{5}$
14. $\frac{5}{6} + \frac{3}{2}$
15. $2.35(4.6)$
16. $\frac{4}{9} - \frac{7}{9}$
17. $\frac{5}{3} + \frac{7}{12}$
18. $\frac{3}{7} - \frac{9}{7}$
19. $\frac{5}{6} - \frac{7}{3}$
20. $\frac{2}{3} + 4$
21. $5 - \frac{3}{7}$
22. $\frac{2}{5} - \frac{3}{20} + \frac{7}{4}$